On Derivations of Basic Governing Equations for Solid Bodies in Bipolar Cylindrical Coordinate System

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Abstract

Traditionally solid mechanics problems are formulated in the Cartesian, cylindrical and spherical coordinate systems. Using such formulation and coordinate systems, solutions of solid mechanics problems are obtained for different geometries such as straight boundary, circular, cylindrical and spherical boundaries. However, the available coordinate systems cannot describe several geometries in spherical or cylindrical coordinate system alone with the inclusion of eccentricities, two spherical or cylindrical bodies in contact and parallel. In this article, author addresses this issue by giving complete original formulations and derivations of basic governing partial differential equations in Bipolar cylindrical coordinate system. The bipolar cylindrical coordinate system allows us to solve several solid mechanics problems such as eccentric cylinders, two parallel cylinders (intersecting or non-intersecting) where the traditional cylindrical or spherical coordinate system cannot handle these problems. Author develops the original equations in the bipolar cylindrical coordinate system in terms of useful quantities of stress and strain components in three dimensions. The paper is limited to development of the basic formulations and practical applications of such formulation can be expanded easily for getting solutions using any known analytical or numerical method.

Keywords: Bipolar cylindrical coordinate system, Elasticity, Deformable solids, Orthogonal curvilinear coordinate system

Introduction

Traditionally, equilibrium problems of solid bodies use only three coordinate systems, i.e., the Cartesian, cylindrical and spherical, to solve any problems. Though, the traditional coordinate systems are capable of solving thousand number of examples accurately, they are insufficient for describing all boundaries and geometries in continuum mechanics problems and real-life practical problems.

By incorporating anew coordinate system, named as the bipolar coordinate system[1], the problems of eccentric cylinders, two parallel intersecting and non-intersection cylinders can be tackled easily wherein, the traditional cylindrical coordinate system cannot handle these problems easily. The references show the basic information about the traditional coordinate system, however, basic governing equations representing the problems are not seen which is very tedious to derive and due to these difficulties, problems having complex geometries and boundaries remain unsolved. In this view, the present paper focuses to use the bipolar cylindrical coordinate system and subsequently derives basic governing equations for tackling static equilibrium problems which are occurring in the bipolar cylindrical coordinate system. This coordinate system is used to tackle problems of eccentric cylinder (Fig.1), two parallel cylinders etc. Few applications of such formulation for transport phenomena are available in ref[2]. Summary list of these non-traditional coordinate systems is seen in references[1,4]. Application areas of non-traditional coordinate systems is seen in few references.

In article[5], author solves a 3-D magnetostatic problem using the cylindrical bipolar coordinate system by separation of variables. The analytical solution applies to the field area between two non-concentric cylindrical surfaces located in close proximity to each other. Rigorous coupled wave analysis of the bipolar cylindrical coordinate system is used to study electromagnetic scattering from an inhomogeneous dielectric material, with eccentric composite circular cylinders are seen in the article[6].A closed-form three-dimensional field solution is presented for cylindrical magnets that are polarized perpendicular to their axes are given in[7]. A dispersive full-wave finite-difference time-domain model is used to study the performance of bipolar cylindrical invisibility cloaking devices, and are given in[8]. Transport phenomena in an eccentric cylindrical coordinate has been studied by[9].A finite-difference method for solving three-dimensional time-dependent incompressible Navier–Stokes equations in an arbitrary curvilinear orthogonal coordinate system is presented in[10]. In above literature, application areas of non-traditional coordinate system in various fields such as electrical and mechanical engineering is used. However, direct use of governing equations for solving solid mechanics problems was also not seen in earlier studies.

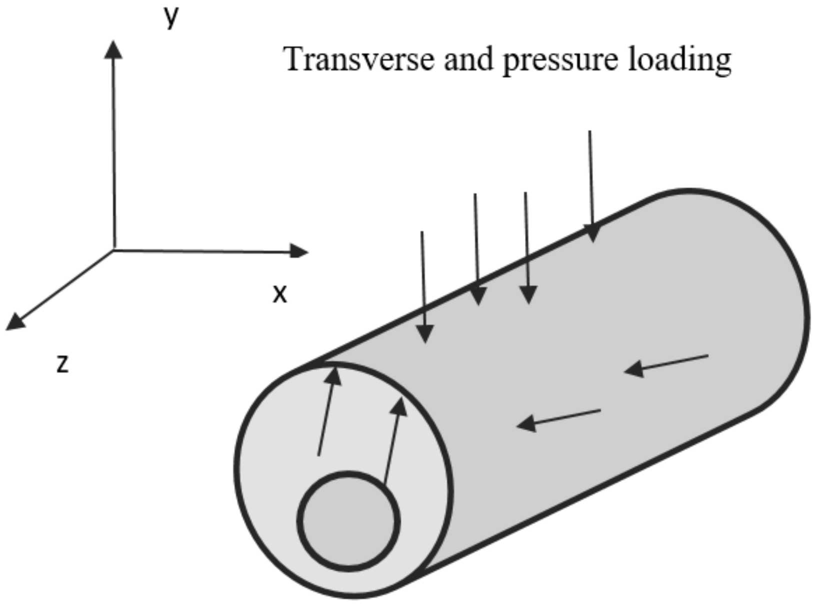


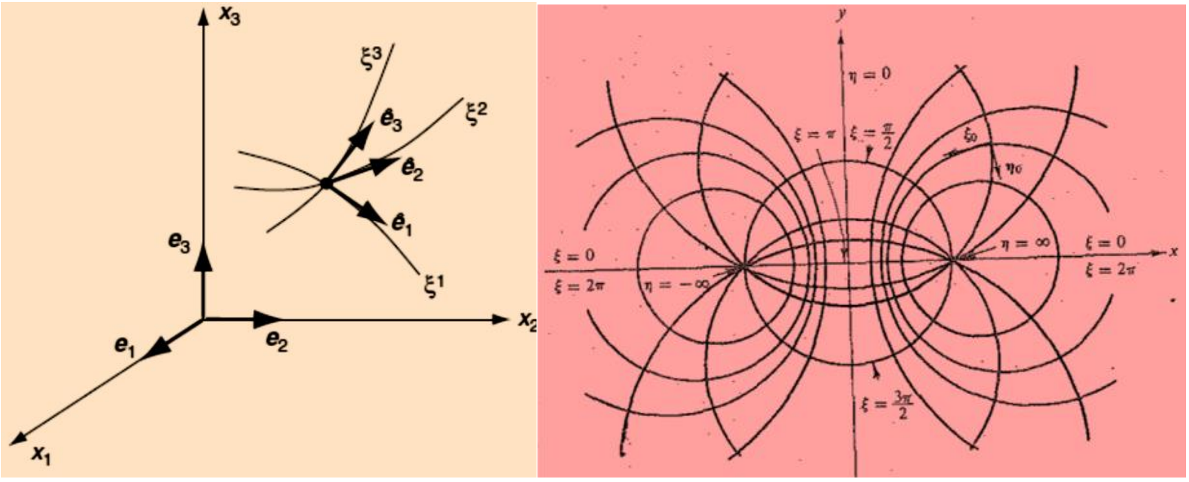
Figure 1: Eccentric 3D cylinder subjected to pressure

Mathematical Formulation

Detailed derivations in Bipolar cylindrical coordinate system

The bipolar cylindrical coordinate system is a three-dimensional orthogonal coordinate system obtained by rotation of two dimensional bipolar cylindrical coordinate about the axis which is joined by two focal points, result in a family of cylinders which are intersecting or non-intersecting with each other. The physical significance regarding representation of the bipolar coordinate system is shown in Fig2b.To develop equations in the system of the bipolar cylindrical coordinate system, the symbols as given in Appendix are used. α, β, z indicates axes of the bipolar cylindrical coordinate system, and(x_1, x_2, x_3)indicates axes of the Cartesian coordinate system. It is required to express the curvilinear basis in terms of the Cartesian basis as follows and symbols are used in accordance with the reference[3] (Fig. 2a).

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial x_k}{\partial \xi^1} e_k, \ \hat{e}_2 = \frac{1}{h_2}\frac{\partial x_k}{\partial \xi^2} e_k, \ \hat{e}_3 = \frac{1}{h_3}\frac{\partial x_k}{\partial \xi^3} e_k \tag{1}$$



(a) **(b)**

Figure 2: Curvilinear coordinate systems and physical significance of bipolar Coordinatesystem[3,4]

Here, k is the number of coordinate axes, i.e., 1, 2, and 3. $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are defined as the unit vectors (in general notation) or $\hat{e}_\alpha, \hat{e}_\beta, \hat{e}_z$ (in bipolar cylindrical coordinate system) are defined as the curvilinear basis in three dimensions. Derivative of this curvilinear basis is required which will be useful eventually while deriving divergence. Also, scale factors are defined as h_1, h_2, h_3 (in general notation) and $h_\alpha = h_\xi, h_\beta = h_\eta, h_z = h_z$ (in the bipolar cylindrical coordinate system) which is an increment in differential length of the element along the curvilinear axis. The coordinates have two foci F_1 and F_2, which are generally taken to be fixed at $(-a, 0)$ and $(a, 0)$, respectively, on the x-axis of the Cartesian coordinate system.

The scale factors for the bipolar coordinate system are given as follows[4],

$$h_1 = h_\alpha = h_\xi = \frac{a}{\cosh\alpha - \cos\beta},\ h_2 = h_\beta = h_\eta = \frac{a}{\cosh\alpha - \cos\beta},\ h_3 = h_z = 1. \tag{2}$$

Also, the relation between the Cartesian and bipolar cylindrical coordinate systems are given by

$$x_1 = x = \frac{a\sin\alpha}{\cosh\alpha - \cos\beta}, \tag{3}$$

$$x_2 = y = \frac{a \sinh \beta}{\cosh \alpha - \cos \beta}, \tag{4}$$

$$x_3 = z = z. \tag{5}$$

To obtaine curvilinear basis as shown in Eq. (1), derivation of the Cartesian coordinate system with respect to the bipolar cylindrical coordinates is carried out using simple use of the quotient rule of differentiation.

Further expressions can be expanded for the derivative with respect to α, β and the third dimension z will give the Curvilinear basis in the form of the bipolar cylindrical coordinate system. It is further followed by the derivation of the curvilinear basis which is necessary to derive the relationship. However, this procedure is quite lengthy, and the following alternative approach as given in Chou and Pagano[11] is used here for the derivation of the curvilinear basis. Derivative of the curvilinear basis is required before deriving the equilibrium equations, derivatives of unit vectors can be given by the following equations as given in Chou and Pagano (1992)[11],

$$\begin{aligned}
&\frac{\partial c_1}{\partial \alpha_1} = -\frac{c_2 \partial h_1}{h_2 \partial \alpha_2} - \frac{c_3 \partial h_1}{h_3 \partial \alpha_3}, \ \frac{\partial c_1}{\partial \alpha_2} = \frac{c_2 \partial h_2}{h_1 \partial \alpha_1}, \ \frac{\partial c_1}{\partial \alpha_3} = \frac{c_3 \partial h_3}{h_1 \partial \alpha_1}, \\
&\frac{\partial c_2}{\partial \alpha_2} = -\frac{c_3 \partial h_2}{h_3 \partial \alpha_3} - \frac{c_1 \partial h_2}{h_1 \partial \alpha_1}, \ \frac{\partial c_2}{\partial \alpha_3} = \frac{c_3 \partial h_3}{h_2 \partial \alpha_2}, \ \frac{\partial c_2}{\partial \alpha_1} = \frac{c_1 \partial h_1}{h_2 \partial \alpha_2}, \\
&\frac{\partial c_3}{\partial \alpha_3} = -\frac{c_1 \partial h_3}{h_1 \partial \alpha_1} - \frac{c_2 \partial h_3}{h_2 \partial \alpha_2}, \ \frac{\partial c_3}{\partial \alpha_1} = \frac{c_1 \partial h_1}{h_3 \partial \alpha_3}, \ \frac{\partial c_3}{\partial \alpha_2} = \frac{c_2 \partial h_2}{h_3 \partial \alpha_3}.
\end{aligned} \tag{6}$$

Here, substituting the values, $c_1 = e_\alpha,\ c_2 = e_\beta,\ c_3 = e_z,\ \alpha_1 = \alpha$ and $\alpha_2 = \beta,\ \alpha_3 = z$ and expressions for scale parameters h_1, h_2,h_3, as given in eq. (2).

$$\frac{\partial e_\alpha}{\partial \alpha} = \frac{\sin\beta}{(\cosh\alpha - \cos\beta)} e_\beta, \ \frac{\partial e_\alpha}{\partial \beta} = \frac{-\sinh\alpha}{(\cosh\alpha - \cos\beta)} e_\beta, \ \frac{\partial e_\alpha}{\partial z} = 0,$$
$$\frac{\partial e_\beta}{\partial \beta} = \frac{\sinh\alpha}{(\cosh\alpha - \cos\beta)} e_\alpha, \ \frac{\partial e_\beta}{\partial z} = 0, \ \frac{\partial e_\beta}{\partial \alpha} = -\frac{e_\alpha \sin\beta}{\cosh\alpha - \cos\beta},$$
$$\frac{\partial e_z}{\partial z} = 0, \ \frac{\partial e_z}{\partial \alpha} = 0, \ \frac{\partial e_z}{\partial \beta} = 0. \tag{7}$$

Eq. (7) are the derivative of unit vectors or the curvilinear basis, which is significantly used further in deriving the general equilibrium equations.

Equilibrium Equations

Generalized expression for a differential divergence operator of a vector field is written as follows[3], where the divergence expression explains how the vector field behaves from one end to another end and a useful quantity to understand the behavior of a solid body subjected to forces.

$$\nabla F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_1 h_3 F_1)}{\partial u_1} + \frac{\partial(h_3 h_1 F_2)}{\partial u_2} + \frac{\partial(h_1 h_3 F_3)}{\partial u_3}\right]. \tag{8}$$

where F_1, F_2 and F_3 are the force components. The values of scale factor h_1, h_2 and h_3, and $(u_1, u_2, u_3) = (\alpha, \beta, z)$ are substituted in this equation. Here, bipolar cylindrical coordinates are in three dimensional form, and hence all three coordinates (α, β, z) will be considered.

$$\nabla F = \left[\frac{(\cosh\alpha - \cos\beta)}{a} . \frac{\partial F_1}{\partial \alpha} - F_1 \frac{\sinh\alpha}{a} + \frac{(\cosh\alpha - \cos\beta)}{a} \frac{\partial F_2}{\partial \beta} - F_2 \frac{\sin\beta}{a} + \frac{\partial F_3}{\partial z}\right]. \tag{9}$$

Defining symbols and now the stress can be expressed in terms of the traction components as

$$\sigma = e_\alpha T_\alpha + e_\beta T_\beta + e_z T_z, \tag{10a}$$

where,

$$F_1 = T_\alpha = \sigma_\alpha e_\alpha + \tau_{\alpha\beta} e_\beta + \tau_{\alpha z} e_z,$$
$$F_2 = T_\beta = \tau_{\alpha\beta} e_\alpha + \sigma_\beta e_\beta + \tau_{\beta z} e_z,$$
$$F_z = T_z = \tau_{\alpha z} e_\alpha + \tau_{\beta z} e_\beta + \sigma_z e_z. \tag{10b}$$

Also Writing,

$$\begin{aligned}
\frac{\partial F_1}{\partial \alpha} &= e_\alpha \left[\frac{\partial \sigma_\alpha}{\partial \alpha} - \frac{\tau_{\alpha\beta} \sin\beta}{\cosh\alpha - \cos\beta} \right] + e_\beta \left[\frac{\sigma_\alpha \sin\beta}{\cosh\alpha - \cos\beta} + \frac{\partial \tau_{\alpha\beta}}{\partial \alpha} \right] + e_z \left[\frac{\partial \tau_{\alpha z}}{\partial \alpha} \right], \\
\frac{\partial F_2}{\partial \beta} &= e_\alpha \left[\frac{\partial \tau_{\alpha\beta}}{\partial \beta} + \frac{\sigma_\beta \sinh\alpha}{\cosh\alpha - \cos\beta} \right] + e_\beta \left[\frac{\partial \sigma_\beta}{\partial \beta} - \frac{\tau_{\alpha\beta} \sinh\alpha}{\cosh\alpha - \cos\beta} \right] + e_z \left[\frac{\partial \tau_{\beta z}}{\partial \beta} \right], \\
\frac{\partial F_3}{\partial z} &= e_\alpha \left[\frac{\partial \tau_{\alpha z}}{\partial z} \right] + e_\beta \left[\frac{\partial \tau_{\beta z}}{\partial z} \right] + e_z \left[\frac{\partial \sigma_z}{\partial z} \right].
\end{aligned} \tag{11}$$

By substituting Eq. (11) into Eq. (8), three terms in Eq. (8) are defined as

1st term of Eq. (8) is given as $\frac{(\cosh\alpha - \cos\beta)}{a} . \frac{\partial F_1}{\partial \alpha} - F_1 \frac{\sinh\alpha}{a}$,

where

$$\begin{aligned}
\frac{(\cosh\alpha - \cos\beta)}{a} . \frac{\partial F_1}{\partial \alpha} - F_1 \frac{\sinh\alpha}{a} &= e_\alpha \left[\frac{(\cosh\alpha - \cos\beta)}{a} \frac{\partial \sigma_\alpha}{\partial \alpha} - \tau_{\alpha\beta} \frac{\sin\beta}{a} - \frac{\sigma_\alpha \sinh\alpha}{a} \right] \\
&+ e_\beta \left[\frac{\sigma_\alpha \sin\beta}{a} + \frac{(\cosh\alpha - \cos\beta)}{a} \frac{\partial \tau_{\alpha\beta}}{\partial \alpha} - \tau_{\alpha\beta} \frac{\sinh\alpha}{a} \right] + e_z \left[\frac{(\cosh\alpha - \cos\beta)}{a} \frac{\partial \tau_{\alpha z}}{\partial \alpha} - \tau_{\alpha z} \frac{\sinh\alpha}{a} \right].
\end{aligned} \tag{12a}$$

2nd term of Eq. (8) is given as $\frac{(\cosh\alpha - \cos\beta)}{a} \frac{\partial F_2}{\partial \beta} - F_2 \frac{\sin\beta}{a}$

where

$$\frac{\left(\cosh\alpha-\cos\beta\right)}{a}\frac{\partial F_2}{\partial\beta}-F_2\frac{\sin\beta}{a}=e_\alpha\left[\frac{\left(\cosh\alpha-\cos\beta\right)}{a}\frac{\partial\tau_{\alpha\beta}}{\partial\beta}+\frac{\sigma_\beta\sinh\alpha}{a}-\tau_{\alpha\beta}\frac{\sin\beta}{a}\right]$$
$$+e_\beta\left[\frac{\left(\cosh\alpha-\cos\beta\right)}{a}\frac{\partial\sigma_\beta}{\partial\beta}-\frac{\tau_{\alpha\beta}\sinh\alpha}{a}-\sigma_\beta\frac{\sin\beta}{a}\right]+e_z\left[\frac{\left(\cosh\alpha-\cos\beta\right)}{a}\frac{\partial\tau_{\beta z}}{\partial\beta}-\frac{\tau_{\beta z}\sin\beta}{a}\right].$$

(12b)

3rd term of Eq. (8) is given as $\frac{\partial F_3}{\partial z}$

which is presented as

$$\frac{\partial F_3}{\partial z}=e_\alpha\left(\frac{\partial\tau_{\alpha z}}{\partial z}\right)+e_\beta\left(\frac{\partial\tau_{\beta z}}{\partial z}\right)+e_z\left(\frac{\partial\sigma_z}{\partial z}\right). \tag{12c}$$

Above three terms are combined together and is rewritten by separating out the terms with the unit vectors, e_α , e_β, and e_z as

$$\nabla F=e_\alpha\begin{bmatrix}\left(\frac{\cosh\alpha-\cos\beta}{a}\right)\frac{\partial\sigma_\alpha}{\partial\alpha}-\tau_{\alpha\beta}\frac{\sin\beta}{a}-\sigma_\alpha\frac{\sinh\alpha}{a}+\left(\frac{\cosh\alpha-\cos\beta}{a}\right)\frac{\partial\tau_{\alpha\beta}}{\partial\beta}\\+\sigma_\beta\frac{\sinh\alpha}{a}-\tau_{\alpha\beta}\frac{\sin\beta}{a}+\frac{\partial\tau_{\alpha z}}{\partial z}\end{bmatrix}$$
$$+e_\beta\begin{bmatrix}\frac{\sigma_\alpha\sin\beta}{a}+\frac{\left(\cosh\alpha-\cos\beta\right)}{a}\frac{\partial\tau_{\alpha\beta}}{\partial\alpha}-\frac{\tau_{\alpha\beta}\sinh\alpha}{a}+\frac{\left(\cosh\alpha-\cos\beta\right)}{a}\frac{\partial\sigma_\beta}{\partial\beta}-\tau_{\alpha\beta}\frac{\sinh\alpha}{a}\\-\sigma_\beta\frac{\sin\beta}{a}+\frac{\partial\tau_{\beta z}}{\partial z}\end{bmatrix}$$
$$+e_z\left[\frac{(\cosh\alpha-\cos\beta)}{a}\frac{\partial\tau_{\alpha z}}{\partial\alpha}-\tau_{\alpha z}\frac{\sinh\alpha}{a}+\frac{(\cosh\alpha-\cos\beta)}{a}\frac{\partial\tau_{\beta z}}{\partial\beta}-\tau_{\beta z}\frac{\sin\beta}{a}+\frac{\partial\sigma_z}{\partial z}\right].$$

(13d)

Further simplification of above equations leads us to the following expressions in the bipolar cylindrical coordinate system,

$$
\begin{aligned}
\nabla F = e_\alpha & \left[\begin{array}{l} \dfrac{(\cosh\alpha-\cos\beta)}{a}\left(\dfrac{\partial\sigma_\alpha}{\partial\alpha}\right)+\dfrac{(\cosh\alpha-\cos\beta)}{a}\dfrac{\partial\tau_{\alpha\beta}}{\partial\beta}+\dfrac{\left(\sigma_\beta-\sigma_\alpha\right)}{a}\sinh\alpha \\ -2\tau_{\alpha\beta}\dfrac{\sin\beta}{a}+\dfrac{\partial\tau_{\alpha z}}{\partial z} \end{array} \right] \\
+e_\beta & \left[\begin{array}{l} \dfrac{(\cosh\alpha-\cos\beta)}{a}\left(\dfrac{\partial\sigma_\beta}{\partial\beta}\right)+\dfrac{(\cosh\alpha-\cos\beta)}{a}\dfrac{\partial\tau_{\alpha\beta}}{\partial\alpha}-\dfrac{\left(\sigma_\beta-\sigma_\alpha\right)}{a}\sin\beta \\ -2\tau_{\alpha\beta}\dfrac{\sinh\alpha}{a}+\dfrac{\partial\tau_{\beta z}}{\partial z} \end{array} \right] \\
+e_z & \left[\begin{array}{l} \dfrac{(\cosh\alpha-\cos\beta)}{a}\left(\dfrac{\partial\tau_{\alpha z}}{\partial\alpha}\right)+\dfrac{(\cosh\alpha-\cos\beta)}{a}\dfrac{\partial\tau_{\beta z}}{\partial\beta}-\tau_{\alpha z}\left(\dfrac{\sinh\alpha}{a}\right) \\ -\tau_{\beta z}\dfrac{\sin\beta}{a}+\dfrac{\partial\sigma_z}{\partial z} \end{array} \right].
\end{aligned}
\tag{14}
$$

Separating the terms inside brackets in eq. (14) will give us the general equilibrium equations in bipolar cylindrical coordinate system in three-dimensional form. The governing partial differential equations in three directions in the bipolar cylindrical coordinate system which are derived here, are readily used for formulation of elasticity problems having eccentricity in cylindrical bodies and helpful in solving the problems along with strain displacement and Hooke’s law. In the following section, strain displacement relations are derived further which are basic essential equations for getting solutions of the cylindrical problems having eccentricity in the geometry. Using these equations, complexity is reduced while assigning the boundary conditions to the posed problem for the bipolar cylindrical coordinate system having cylindrical eccentricity.

$$
\begin{aligned}
&\frac{(\cosh\alpha-\cos\beta)}{a}\left(\frac{\partial\sigma_\alpha}{\partial\alpha}\right)+\frac{(\cosh\alpha-\cos\beta)}{a}\frac{\partial\tau_{\alpha\beta}}{\partial\beta}+\frac{\left(\sigma_\beta-\sigma_\alpha\right)}{a}\sinh\alpha-2\tau_{\alpha\beta}\frac{\sin\beta}{a}+\frac{\partial\tau_{\alpha z}}{\partial z}=0,\\
&\frac{(\cosh\alpha-\cos\beta)}{a}\left(\frac{\partial\sigma_\beta}{\partial\beta}\right)+\frac{(\cosh\alpha-\cos\beta)}{a}\frac{\partial\tau_{\alpha\beta}}{\partial\alpha}-\frac{\left(\sigma_\beta-\sigma_\alpha\right)}{a}\sin\beta-2\tau_{\alpha\beta}\frac{\sinh\alpha}{a}+\frac{\partial\tau_{\beta z}}{\partial z}=0,\\
&\frac{(\cosh\alpha-\cos\beta)}{a}\left(\frac{\partial\tau_{\alpha z}}{\partial\alpha}\right)+\frac{(\cosh\alpha-\cos\beta)}{a}\frac{\partial\tau_{\beta z}}{\partial\beta}-\tau_{\alpha z}\left(\frac{\sinh\alpha}{a}\right)-\tau_{\beta z}\frac{\sin\beta}{a}+\frac{\partial\sigma_z}{\partial z}=0.
\end{aligned}
\tag{15}
$$

Strain Displacement Equations

In solid body mechanics, gradient is the strain-rate tensor which is a physical quantity that describes the rate of change of the deformation of a material in the neighborhood of a certain point. It is a useful quantity which represents the relationship between strain and geometrical deformation of the solid body. Generalized form of gradient of ϕ is expressed as follows,

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial\alpha_1}c_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial\alpha_2}c_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial\alpha_3}c_3. \tag{16a}$$

Furthermore, the displacement vector u is defined based on its three directional component as follows,

$$\phi = u = u_\alpha e_\alpha + u_\beta e_\beta + u_z e_z. \tag{16b}$$

The strain e is a symmetric second-order tensor and is commonly written in matrix format as

$$e = \begin{bmatrix} u_{\alpha\alpha} & u_{\alpha\beta} & u_{\alpha z} \\ u_{\beta\alpha} & u_{\beta\beta} & u_{\beta z} \\ u_{z\alpha} & u_{z\beta} & u_{zz} \end{bmatrix}. \tag{16c}$$

Here, we define three normal and three shearing strain components leading to a total of six independent components that completely describe small deformation theory. Here, $u_{\alpha\alpha}$, $u_{\beta\beta}$ and u_{zz} are the normal strain and remaining quantities of the matrix are shear strain. This set of equations is normally referred to as the strain-displacement relations which are derived using curvilinear coordinate system in the bipolar cylindrical coordinate system. The development of the strain-displacement relations in the bipolar cylindrical coordinate system is given as follows. First, substituting the values as $\alpha_1 = \alpha,\ c_1 = e_\alpha,\ \alpha_2 = \beta,\ c_2 = e_\beta,\ \alpha_3 = z,\ c_3 = e_z$ and scale parameters from Eq. (2) into Eq. (16a).

Upon substituting the above equations of strain and strain tensor in Eq. 16b-16c, and performing the derivative operations, we get following equations,

$$
\begin{aligned}
\nabla\phi &= \frac{1}{h_1}\frac{\partial\phi}{\partial\alpha_1}c_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial\alpha_2}c_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial\alpha_3}c_3, \\
\nabla\phi &= \frac{(\cosh\alpha-\cos\beta)}{a}\frac{\partial\phi}{\partial\alpha}e_\alpha + \frac{(\cosh\alpha-\cos\beta)}{a}\frac{\partial\phi}{\partial\beta}e_\beta + \frac{\partial\phi}{\partial z}e_z.
\end{aligned} \tag{17}
$$

Substitution of the displacement vector from Eq. (16a) and replacing the generalized symbols from 1, 2 and 3 with $\alpha,\ \beta,\ z$, the right hand side of first term of Eq. (17) is given as follows.

$$
\begin{aligned}
\frac{\partial\phi}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\left[u_\alpha e_\alpha + u_\beta e_\beta + u_z e_z\right] \\
&= e_\alpha\frac{\partial u_\alpha}{\partial\alpha} + u_\alpha\frac{\partial e_\alpha}{\partial\alpha} + e_\beta\frac{\partial u_\beta}{\partial\alpha} + u_\beta\frac{\partial e_\beta}{\partial\alpha} + e_z\frac{\partial u_z}{\partial\alpha} + u_z\frac{\partial e_z}{\partial\alpha}
\end{aligned} \tag{18}
$$

Upon substitution of derivatives of the unit vectors appears in above equations and simplifying the equations further,

$$
\begin{aligned}
\frac{\partial\phi}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\left[u_\alpha e_\alpha + u_\beta e_\beta + u_z e_z\right] \\
&= e_\alpha\frac{\partial u_\alpha}{\partial\alpha} + u_\alpha\frac{\partial e_\alpha}{\partial\alpha} + e_\beta\frac{\partial u_\beta}{\partial\alpha} + u_\beta\frac{\partial e_\beta}{\partial\alpha} + e_z\frac{\partial u_z}{\partial\alpha} + u_z\frac{\partial e_z}{\partial\alpha} \\
&= e_\alpha\frac{\partial u_\alpha}{\partial\alpha} + u_\alpha\frac{e_\beta\sin\beta}{\left(\cosh\alpha-\cos\beta\right)} + e_\beta\frac{\partial u_\beta}{\partial\alpha} - u_\beta\frac{e_\alpha\sin\beta}{\left(\cosh\alpha-\cos\beta\right)} + e_z\frac{\partial u_z}{\partial\alpha} + 0 \\
&= e_\alpha\left[\frac{\partial u_\alpha}{\partial\alpha} - \frac{u_\beta\sin\beta}{\left(\cosh\alpha-\cos\beta\right)}\right] + e_\beta\left[\frac{\partial u_\beta}{\partial\alpha} + \frac{u_\alpha\sin\beta}{\left(\cosh\alpha-\cos\beta\right)}\right] + e_z\frac{\partial u_z}{\partial\alpha}.
\end{aligned} \tag{19}
$$

Similarly, second and third terms are obtained. Collecting all the terms together, Eq (14) presents as follows by separating out the terms with the unit vectors e_α, e_β and e_z in the bipolar cylindrical coordinate system,

$$
\phi = \nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial\alpha_1}c_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial\alpha_2}c_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial\alpha_3}c_3. \tag{20a}
$$

$$
\begin{aligned}
&= e_\alpha e_\alpha \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\alpha}{\partial \alpha} - \frac{u_\beta \sin\beta}{a} \right] + e_\beta e_\beta \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\beta}{\partial \beta} - \frac{u_\alpha \sinh\alpha}{a} \right] \\
&+ e_z e_z \left[\frac{\partial u_z}{\partial z} \right] + e_\alpha e_\beta \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\beta}{\partial \alpha} + \frac{u_\alpha \sin\beta}{a} + \frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\alpha}{\partial \beta} + \frac{u_\beta \sinh\alpha}{a} \right] \\
&+ e_\beta e_z \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_z}{\partial \beta} + \frac{\partial u_\beta}{\partial z} \right] + e_\alpha e_z \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_z}{\partial \alpha} + \frac{\partial u_\alpha}{\partial z} \right]
\end{aligned}
\tag{20b}
$$

In Eq. (20), the individual bracket terms give the desired strain displacement relations as given in matrix Eq. (17b) in the form of the bipolar cylindrical coordinate system. The individual scalar equations are given by

$$
\begin{aligned}
u_{\alpha\alpha} &= \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\alpha}{\partial \alpha} - \frac{u_\beta \sin\beta}{a} \right], \\
u_{\beta\beta} &= \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\beta}{\partial \beta} - \frac{u_\alpha \sinh\alpha}{a} \right], \\
u_{zz} &= \left[\frac{\partial u_z}{\partial z} \right], \\
u_{\alpha\beta} &= \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\beta}{\partial \alpha} + \frac{u_\alpha \sin\beta}{a} + \frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_\alpha}{\partial \beta} + \frac{u_\beta \sinh\alpha}{a} \right], \\
u_{\beta z} &= \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_z}{\partial \beta} + \frac{\partial u_\beta}{\partial z} \right], \\
u_{\alpha z} &= \left[\frac{(\cosh\alpha - \cos\beta)}{a}\frac{\partial u_z}{\partial \alpha} + \frac{\partial u_\alpha}{\partial z} \right].
\end{aligned}
\tag{21}
$$

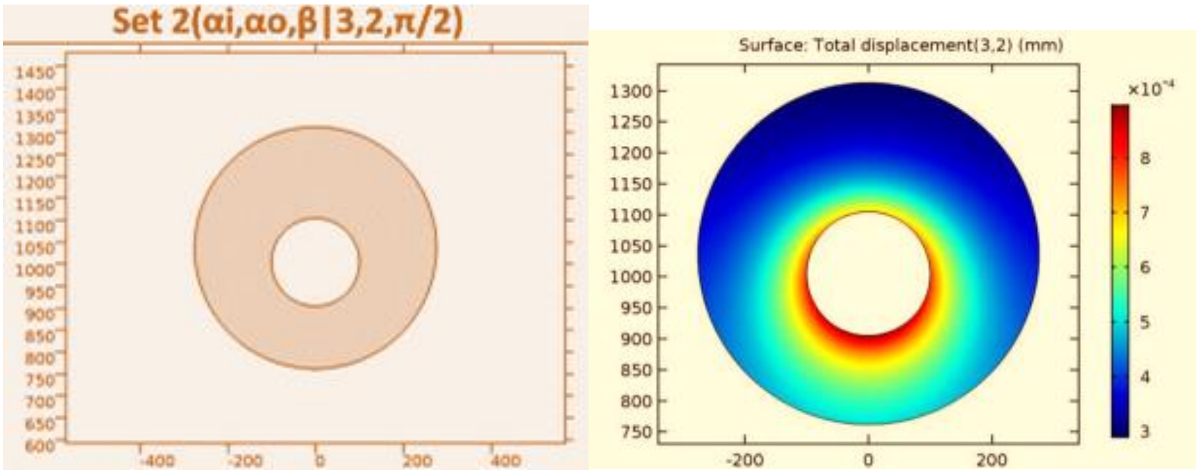
The above equations are the strain-displacement relationship in the bipolar cylindrical coordinate system in the three-dimensional form. The strain-displacement relationship is used to determine the strain in the solid body. Eq. (21) are used for special problems formulated in the bipolar cylindrical coordinates and analyzed using analytical or numerical method.

Numerical illustration and usefulness of the proposed study

Few examples of the real-life applications are illustrated here considering the prospect applications of the derived equations in this article.

Example 1 – Civil and Structural engineering applications

Consider the solution of two-dimensional plane stress problem of ring with eccentricity through either analytical or numerical solutions to investigate the parameters adversely effect on the stress and strain due to eccentricity. In real life engineering such applications are seen in infinite length pipelines where walls of the pipelines are weakened through corrosion or environmental effect and reducing the load carrying capacity of the structure. It is necessary to investigate the stresses and strain in the weakened structure help engineers to understand the defect and results are useful for the future design. Current equations have the potential applications for tackling such problems. Fig. 3 shows the geometrical and output parameters which are the result of such formulations.



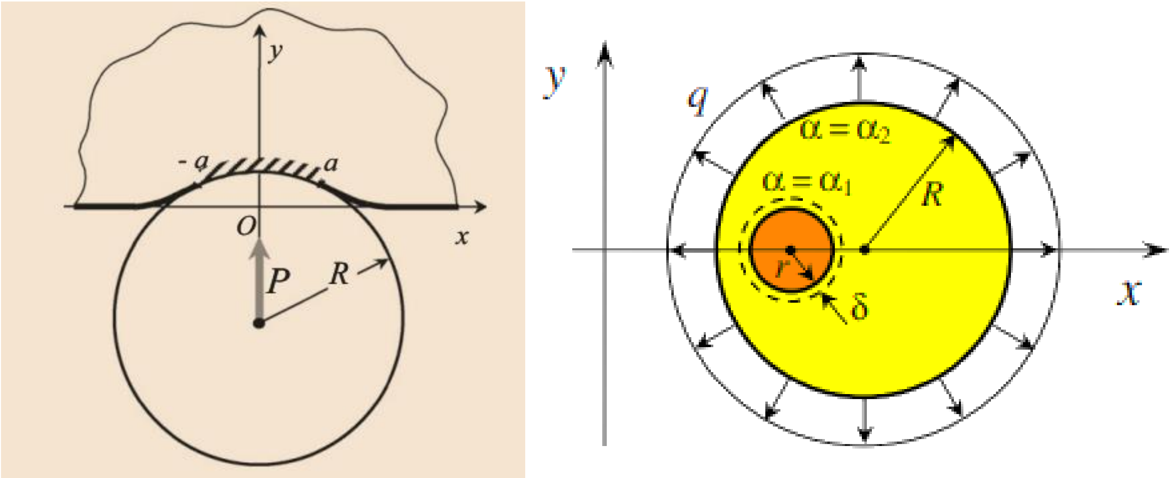
(a) (b)

Figure 3: 2D ring problem having eccentricity subjected to pressure

Example 2 – Mechanical engineering applications

Present equations can be formulated for the for the application of Interaction between ideal surfaces in contact characterized by acted forces in such a way lead to the so-called equilibrium separation and adhesiveness contact between two surfaces are investigated as areas of application of contact mechanics and useful in design of adhesive surfaces and manufacturing of micro devices. Representation of the problem of adhesive contact of a rigid cylinder with an elastic half-space is shown in fig.

Analytical problem or numerical treatment of such problem is formulated in bipolar coordinates to evaluate the shear stresses between the contacting surfaces by assuming contact with friction. Such problems have an immense engineering applications, present formulation facilitates for deriving the governing equations to present mechanics of contact problems



(a) (b)

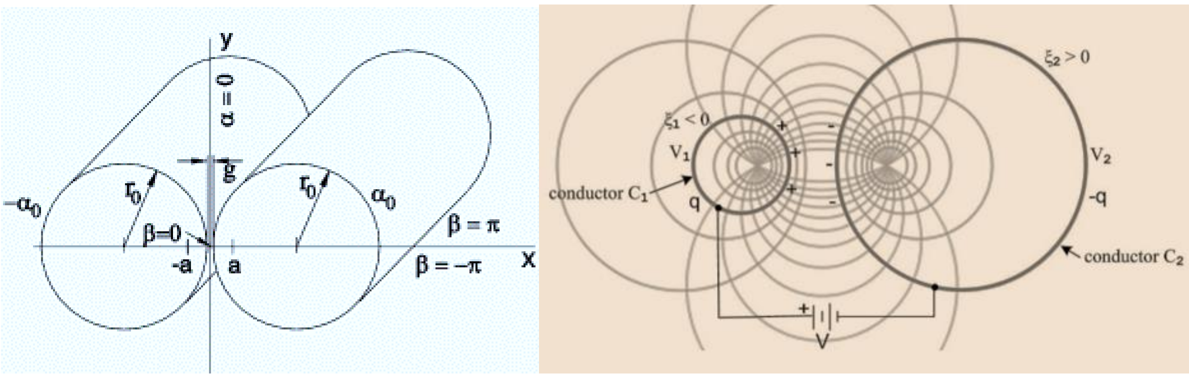
Figure 4: (a) Contact of an elastic half space with a rigid cylinder[12] and (b) Eccentric circular inclusion of press -fitted disk[13]

Example illustration 3 - Mechanical engineering applications

Another real-life engineering problem which can be tackled using the present formulation is to investigate the stresses induced in an elastic and isotropic disk by an eccentric press-fitted circular inclusion. Representation of such problem is shown in Fig. 4 and given in reference[12]. However, in this reference, tedious mathematical technique is used which can be replaced with use of convenient equations derived in the present work formulation. The disk is also subject to uniform normal stress applied at its outer border. The inclusion is assumed to be of the same material as the annular disk and both elements are in a plane stress or plane strain state. A frictionless contact condition is assumed between the two members.

Example illustration 4 – Electrical engineering applications

Apart from above illustration from mechanics, also real engineering application in electrical engineering is seen where magnetic field calculation of tubular linear permanent-magnet machine assemblies is required to calculate, where, off centered rotor and for the analysis of the field of a two – roller magnetizing fixture is formulated in cylindrical bipolar coordinates. Fig. 6 (a) represents schematic of this problem.



(a) (b)

Figure: 6 (a) Tubular Linear Permanent-Magnet Machine Assemblies[5] and (b)The electrostatic problems with parallel cylinders[14]

Example illustration 5 - Electrical engineering applications

The electrostatics problem of two infinite, parallel, conducting cylinders provides a real-world application of the use of bipolar cylindrical coordinate system. The schematic image reflecting the problem is to find the capacitance per unit length C is shown in Fig. 6(b).

Conclusion

In this paper, author presented an original and detailed systematic derivations of governing partial differential equations in the bipolar cylindrical coordinate system comprising of a stress and strain tensor. A governing differential equation and physical quantities of strain are derived based on the following factors which represent the exact theory of elasticity in the bipolar cylindrical coordinate system, as (1) the orthogonal curvilinear coordinate systems, (2) scale factors, (3) Cartesian coordinate system in terms of the bipolar cylindrical coordinates and (4) the curvilinear basis in terms of the Cartesian basis. The formulations of the curvilinear basis in terms of Cartesian basis and its derivatives are formulated in the bipolar cylindrical coordinate systems in the three-dimensional form. Equations are very useful in tackling several static equilibrium problems of solid cylindrical bodies which cannot be tackled by using traditional cylindrical coordinate system. The equations are generalized, and can conveniently be used for future numerical and analytical solutions in various real life practical solid mechanics problems such as eccentric hole in the three-dimensional solid cylinder, elastic half space with cylindrical geometry, family of interacting cylindrical body subjected to various loadings and boundary conditions and parallel cylinders. Such configurations are used in engineering structures such as pressure vessels, pipe lines, aircraft body and various parts of the machines. Real life applications and usefulness of the formulations of the proposed study is discussed with illustrations in the article.

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Appendix: Mathematical symbols used in the formulation of the bipolar cylindrical coordinate System

α, β, z	Coordinate in the Bipolar cylindrical Coordinate System
x_1, x_2, x_3	Direction in Cartesian coordinates
h_α, h_β, h_z	Scale factors in the bipolar cylindrical coordinate system
h_1, h_2, h_3	Scale factors in the generalized system
$\hat{e}_1, \hat{e}_2, \hat{e}_3$	Curvilinear basis in generalized symbol
$\hat{e}_\alpha, \hat{e}_\beta, \hat{e}_z$	Curvilinear basis in generalized symbol
F_1, F_2 and F_3	components of the Traction in generalized symbol
T_α, T_β, T_z	Components of traction
ξ^1, ξ^2, ξ^3	orthogonal curvilinear coordinates
$\sigma_\alpha, \tau_{\alpha\beta}, \tau_{\alpha z}$ $\sigma_\beta, \tau_{\alpha\beta}, \tau_{\beta z}$ $\sigma_z, \tau_{\alpha z}, \tau_{\beta z}$	Stress tensor in the bipolar cylindrical coordinate system
$u_{\alpha\alpha}, u_{\alpha\beta}, u_{\alpha z}, u_{\beta\beta}, u_{zz}, u_{\beta z}$	Normal and shearing Strain in the bipolar cylindrical coordinate system

Glossary

Rigid body- Body that does not deform or change its shape

Solid body- Body that deforms or change its shape

Solid mechanics- It is the branch of mechanics that studies the behaviour of solids under the action of external forces and displacement.

Behaviour of solid- Behaviour of solid is studied to evaluate the critical parameters such as deformations, stresses, strains in the solid body which are helpful for engineering design

Coordinate system- It is the number coordinates uniquely define the position of a particle.

Ordinary differential equations- Governing equations and mathematical expressions in one dimension (1D) that represents the physical behaviour of solids (in the present case) contains the unknown critical parameters such as deflections, stresses, strains and are required to solve using mathematical techniques either, numerical or analytical

Partial differential equations- Governing equations and mathematical expressions in two dimensional (2D) or three-dimensional domain (3D) that represents the physical behaviour of solids (in the present case) contains the unknown critical parameters such as deflections, stresses, strains and are required to solve using mathematical techniques either, numerical or analytical

Curvilinear coordinate system- It is the coordinate system where coordinate lines are curved

Scale factors- An increment in differential length of the element along the curvilinear axis

Stress- It is the intensity of the measured per unit cross sectional area assumed to be distributed over the area approximately in approximate theory and assumed to be acted at a point in exact theory of elasticity

Strain- Strain occurs in solid body as a result of stress due to external forces

Stress equilibrium equations- They are ordinary or partial differential equations presenting the unknown quantities as stresses and its change from one section to another section

Divergence- It is similar to stress equilibrium equations engineering derived from free body diagram of forces, however, in vector calculus and curvilinear coordinate system it is termed as divergence

Strain-displacement relations- It is a measure of how rapid the displacement changes through the material

Gradient- It is similar to strain-displacement relations in engineering which is derived from free body diagram of forces, however, in vector calculus and curvilinear coordinate system it is termed as gradient

Normal stress- Stress acting in normal (perpendicular) direction

Normal strain- Strain acting in normal (perpendicular) direction

Shear stress- Stress acting in tangential direction

Shearing strain- Strain acting in tangential direction

Unit vector or Cartesian basis- Vector whose unit is unit

Tensor- Tensor is defined by the number of directions

Vector- Vector is first order tensor, means it has a single direction