Mathematics Teaching - Balancing Abstract Verses Concrete

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My forms are not abstractions of things in the real world. They're also not symbols. I would say that my job is to invent these forms and to put them together in a way that keeps your interest, to give the forms a quirky identity so you can engage with them, so you realize there's an inner intelligence or logic.

Caio Fonseca (American Painter)

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Abstract

Mathematics educators all over world find it difficult to the engage learners towards the effective learning of mathematical concepts. Interiorizing these abstract ideas without referring to anything real or tangible, require thinking and visualizing skills at higher cognitive level. To overcome this difficulty, real life examples were made mandatory for mathematical teaching, which helped in improving the understanding level to some extent. However, associating all mathematical concepts to things which the students can see or know was not only difficult but also failed to develop the skills required to generalize these concepts after decontextualizing them from these base cases. The present work is based upon a survey of the methods adopted in mathematics education at various levels and also discusses the effectiveness of teaching Linear algebra to undergraduate students in an abstract way with very limited but appropriate usage of concrete examples.

Keywords

Abstraction, Generalization, Decontextualisation

Introduction

Teaching mathematics at all levels of education remains challenging for instructors. On one hand, it is highly satisfying to teach a handful of motivated students who enjoy exploring the ultimate truth in the supreme beauty of mathematics, while on the other, it is frustrating to see most students struggling to understand the concepts and apply them to the methods employed for finding solutions to contextualized problems.

Various studies[1-3] over the past several decades have attributed the '*mathematics phobia*' among young learners to the '*abstractness in mathematics*'. It is interesting to see that largely students believe that they learn mathematics as a part of the curriculum which has no relevance to anything concrete in real life situations. Everyday words with meaning in relation to other mathematical terms (without their actual meanings) are used, say for example '*zero*' which means '*nothing*' in real world, is understood as the '*additive identity*', representing different things in different mathematical structures. In Vector algebra it represents a vector with zero magnitude and no direction, whereas in quotient group of integers modulo n (n is an integer), the number n and all its multiples are equivalent to zero. Thus in the quotient group of integers modulo 2, all even numbers are identified with the zero, a concept very different from the meaning of *zero* used elsewhere.

Owing to the abstractness in mathematics and mathematical learning, serious attempts are made by instructors to make mathematics meaningful by linking the ideas to concrete materials or find their real life applications. In the last few decades, teaching methods involving concrete examples were emphasized to such an extent that the journal '*Mathematics Teacher*' (November 1993) devoted the entire issue to this aspect and the editorial board challenged teachers to find at least one example or a link for each topic in mathematics they taught with a real life situation. However, an important research[4] in this area has shown that relating concrete examples or tangible things to abstract mathematical ideas failed to enhance the learning among the students substantially. This can be understood if we take the example of teaching the concept of fractions to elementary school students. Teachers conventionally give example of pieces of pizza or chalk *etc.* to link the concept with real life objects around them. Students at this level are already familiar with counting whole numbers. Although the number system was introduced to them with concrete examples of objects such as apples or counting fingers etc., over the years they have learnt to see, say, '5' as a number detaching it from these examples. They know that 3 can be subtracted from 5 but it is not possible to take away 3 oranges from 5 apples. Thus, subtraction becomes an object of thought for them and the operation is performed at an upper cognitive level without referring to any particular example. Skemp[8] noted that this was the unique ability of human beings to isolate concepts from any of the examples giving rise to them. However, it was seen that this ability to isolate the concept from the base context took a couple of years after learning the concepts. Thus, a similar approach of decontextualizing mathematical operations on fractions fails to reduce the complexity of the problems, as students neither see fractions as pieces of pizzas always nor do they realize them as numbers when they operate upon them. Thus a child identifies *one sixth* of a pizza as *one piece* of pizza without referring to the whole pizza at all. Consequently, two pieces (one sixth each) of a pizza will not be seen as one third of the pizza by him. With a heavy emphasis on computational methods in elementary classes, students learn to manipulate problems involving fractions, but to convert the concept into an object of thought appears difficult.

Approaches in Mathematics education

Mathematics educators are often blinded with their assumption that generalization of a concept is a trivial process and it comes naturally to the learners. Some believe that mathematics should be taught as concrete $\longrightarrow$ abstract, which means starting with real world manipulative to understand the underlying principal first. The concept of differentiation at undergraduate level is introduced with a whole lot of examples say, change in velocity, growth of bacteria and economic growth, etc. followed by the classical definition of derivatives. Another approach of teaching mathematics is from abstract $\longrightarrow$ concrete. In fact most of the mathematics at the middle school and high school level is taught with the *abstract first* approach. For example, polynomials are taught without referring to their applicability to anything in the real world. Interestingly, polynomials are considered to be the building blocks of all sciences because of being simple curves which can be easily manipulated, but capable of approximating complicated curves associated with all changes in nature such as weather, path of planets, mechanical forces, chemical and biological processes, etc. But, either the educators are unaware of the importance of polynomials in the study of science and technology at advanced levels or fail to communicate their importance to the students restricting themselves to the prescribed syllabus. Hence, students at high school level learn polynomials purely as symbolic expressions with the techniques of finding roots and factors without knowing about the significance of the concepts.

It was observed that both of the aforesaid pedagogical approaches have limitations as far as learning is concerned. Students learning with *concrete first* approach remain grounded towards the examples discussed and are not able to generalize the concepts for application to a wider variety of mathematical structures. On the other hand those who learn from *abstract first* approach lose interest because of the abstractness associated with mathematics and its education. Nevertheless, in both of these approaches of teaching-learning mathematics, the concept of abstractness is understood as something which is not related to the real world. But as we see in the next section, abstractness in mathematics is also associated to a general idea perceived from a collection of apparently different concepts.

What is abstractness in mathematics?

In order to understand the 'abstractness' ascribed to mathematics we need to first understand the word '*abstract*'. According to the Webster Dictionary (1977) we have:

Abstract (adj.)**:** Apart from concrete; general as opposed to particular; expressed without reference to particular examples (e.g. numbers).
Abstract (verb)**:** To consider apart from particular instances; to form a general notion of.

It is interesting to see that although abstractness of mathematics is understood as it's disconnectedness from anything concrete in real life, but from the definitions it appears that '*apartness from concrete*' and '*generality*' are the two aspects to which it is attached. Mitchelmore et al.[5-7] have classified mathematical ideas into two categories, *abstract-apart* and *abstract-general.* The ideas which are *abstract apart* are actually disconnected from any particular context and hence need not be applicable to real life situation. On the other hand *abstract general* ideas are those ideas which arise in the process of abstracting a common property from a variety of disconnected base contexts, after recognizing some similarity between them. For example, before teaching the methods of solving linear equations symbolically, teachers give a series of situations in real life which give rise to such equations. These *abstract general* ideas help the learner to build up new relations at higher level of abstraction in terms of already familiar base contexts and allow them to develop ideas to be applied to relatively newer situations. At this level of understanding the existing ideas become more general and hence more abstract.

Abstraction as a component of mathematics education

Thus, mathematical abstraction is more of a process than a product and it should be well integrated in mathematics education. If the focus of mathematics education is more on pattern recognition, which means finding similarities between apparently different structures, mathematics becomes more appealing to the students. They learn to develop relations and identify the set of rules applicable to equivalent systems. Contrarily, in most of the situations we see that mathematics education is imparted as transfer of *abstract apart* ideas without going through the process of abstraction. Even though some students enhance their computational skills by mastering the rules to work upon the symbolic representations, but majority of them lose interest in mathematics thinking it to be all about boring calculations guided by meaningless set of rules. Algebra, one of the most important branches of mathematics is taught in a way which has very little to cater to the thought process of developing the algebraic expressions. As an example, learning the techniques of finding roots to complicated algebraic equations is given more importance than to understand the geometric interpretation and the behavior of the curves in the neighborhood of these roots. Only some students who are so called '*mathematically abled*' learn how to simplify algebraic expressions without thinking or knowing much about the implications of the methods involved. But, not much is gained in terms of mathematical insight because with the intervention of computers, computational skill is anyway a lost battle to the present day software available. Hence, learning mathematics necessarily requires developing the skill to think beyond the contexts along with the ability to visualize the geometry, recognize pattern or understand the deeper meaning associated with the concepts.

Limiting over-usage of concrete examples in mathematics education

Furthermore, there are conflicts regarding the usage of concrete examples by mathematics instructors. A very important finding of Kaminski[4] was that although these examples help in understanding the abstract mathematical ideas by relating them with things we know about, but keep us from the generation of newer ideas or application of the existing ones to novel situations. On the other hand, playing around with pure mathematical symbols makes the very essence of mathematics impregnable for the learner. In fact, a careful study of teaching methodologies revealed the fact that in mathematics education, there is a dichotomy in both the approaches. We have seen that in the last decade an enormous number of research papers have been published supporting either of the two approaches strongly. However, we tried to experiment with a different pedagogy for teaching mathematical concepts at undergraduate level, at Navrachana University, Vadodara, by introducing the abstract concepts directly without relating to anything concrete, in the introductory lectures. We observed that initially the students struggled to interiorize the concepts in the absence of examples which were easy to visualize, but gradually they learnt to cope with the abstractness of mathematics. In fact, as an extension of theoretical learning, they started relating the abstract mathematical ideas with things they could envisage. Also, as every brain perceives things in a unique way, they were discussing a larger variety of situations which could be associated to the understanding of the concepts. It was like writing a story about an abstract piece of art work depending upon each one's own perception. We saw that the inhibitions were broken and everyone participated in the teaching-learning process which is very rare in a mathematics class. We tried to disengage mathematics learning with the idea of learning problem solving techniques alone. In fact, as instructors we ourselves learnt that getting wrong answer was the greatest fear among the learners, holding them back from trying out unknown methods to get the solution. We rated understanding, thinking and trying out newer approaches towards the solution higher than applying text book methods to come to the correct solution. Abstractness in mathematics no more appeared to be an obstacle in learning because it stimulated thinking and visualization, which is a natural play area of every young brain.

In order to explore more in this area, we studied the effect of establishing smooth bridges between both the approaches involving abstract and concrete, so that one can gently pass from one approach to the other depending upon the depth of the mathematical ideas and requirement of the learners. For example, an average engineer is required to know more about the applicability of the ideas to different situations rather than exploring deep into the logical flow and intricate mathematical steps used in proving them. However, the fast changing world today also requires them to understand the mathematical concepts to a greater depth, to cope with and contribute towards the continuously emerging technologies.

Teaching Linear Algebra in abstract way

We discuss the pedagogy involved in the teaching of Linear Algebra at undergraduate level in an abstract way in this section. Linear algebra is an advanced course in mathematics, introduced to all engineering undergraduate students, as it finds wide applications in every field of science and technology. However, students universally find it difficult to understand, presumably due to the transition from elementary to advanced mathematics. In most of the cases, the course is taught by the *concrete first* approach and builds upon the matrix theory, with which the students are already familiar. In this method of *concrete to abstract* teaching, the emphasis lies in dealing with the computations involving matrices, at the introductory level of the course. Now, the heart of the subject lies in interiorizing the variety of structures as vector spaces from the familiar two dimensional Euclidean plane R^2 to more abstract spaces such as the space of polynomials or the space of continuous functions. We observed that the course starting with examples of R^2 and R^3 actually impedes the process of learning, keeping learners stuck with the calculations involved in proving the axioms and operating upon the matrices. The algorithms applicable to more abstract vector spaces are very different in nature to these operations and students taught with *concrete first* approach do not learn to visualize such situations and to apply appropriate algorithms to deal with them. For example, finding a basis for a vector space of continuous functions require very different algorithm than finding a basis for row vectors in $R^2$and R^3. Instead, we saw that by taking the *abstract first* approach and directly introducing the abstractness in the concept of vector spaces, a majority of students could conceptualize the ideas in the most general settings. The computational techniques were introduced as and when needed, focusing more on the identification of the deeper similarities between wide varieties of mathematical structures. This method of *abstract first* learning supported by suitable concrete examples provided an environment where the students were motivated to become independent learners and thinkers in a shorter span.

Conclusions

Our conclusion is that the abstract mathematical ideas should be appropriately balanced with concrete examples in mathematics education. Educators need to be aware of the extent of abstractness associated with individual topics within the courses and select concrete examples which are just enough to keep the learners engaged with the course. Although mathematics teaching cannot be completely devoid of concrete examples, but their usage should be limited to introductory levels only. Mathematics is abstract and sooner the students are exposed to the abstractness and the process of abstraction, greater is their achievement in terms of appreciating and understanding the concepts. There is an incessant list of examples where the ideas which were once considered highly abstract at some point, found applications in real world later. With appropriate choice of such examples students should be made aware of the fact that the beauty of mathematics and it's applicability are two different aspects but learning mathematics with all its abstractness widens the scope of finding applications, along with opening up the heart, mind and soul towards the subject, which is very essential for its effective learning.

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