Direct Stiffness Method for a Curved Beam and Analysis of a Curved Beam Using SAP

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Abstract

Analysis of curved beam has been carried out for the unknown displacements and rotations subjected to horizontal and vertical loads and the moments at the ends. The concept of strain energy and Castigliano's theorem has been used to carry out the analysis. Two noded curved beam elements have 3 degrees of freedom (1) Rotation in out of plane direction, (2) Horizontal displacement and (3) Vertical displacement. Through energy concept, force - displacement relationship has been established for which 9 equations (3 for vertical displacements, 3 for rotations and 3 for horizontal displacements) are obtained by keeping node 1 fixed of the curved beam and other 9 equations (3 for vertical displacements, 3 for rotations and 3 for horizontal displacements) are obtained by keeping node 2 fixed. Remaining 18 equations are obtained from the equilibrium equations. These equations are used to form the element stiffness matrix. Final element stiffness matrix turns out to be of the size 6 x 6. Numerical example of the curved beam is analysed and for which the results are compared with the SAP2000.

Keywords

Direct Stiffness method, curved beams, Strain energy and Castigliano's Theorem

Introduction

Curved Beam is an elastic body whose geometric shape is formed by the motion in space of a plane figure called the cross section of the curved beam. The cross section of the curved beam is normal to the axis of the curve. Curved beam is a special case of an arch where the radius is constant. Moreover, this type of beam is used where a beam having long span without any intermediate support is to be provided. Few analytical formulations related to curved beam structural analysis has been done in the past. The derivation of generalized stiffness matrix of a curved – beam element including normal - to – plane force components has been obtained by[1].

Based on the theory of virtual work and principle of thermal elasticity, exact solutions for in – plane displacements of curved beams with pinned – pinned ends are derived explicitly by [2]. A curved beam element for its application to traffic poles with assumptions curved traffic poles may vibrate when excited by the wind[3]. A 3 node curved beam element has been developed for static and dynamic analysis of curved poles. Shear deformation as well as rotary inertia effects are included. A stiffness matrix of order 12 x 12, for a curved beam element involving all forces together using Castigliano's theorem including the effects of transverse shear forces and tangential thrust given by[4].
When the structure is linear elastic and the deformations are caused by external forcesonly, the strain energy U, the displacement of structure in the direction of force is expressed by [5, 6]

$$\Delta_j = \frac{\partial U}{\partial P_j} \tag{1}$$

The Castigliano's theorem states that having the expressions for the strain energy invarious cases, avery simple method for calculating the displacements of point of an elastic body duringdeformation may be established. For example, in the case of simple tension

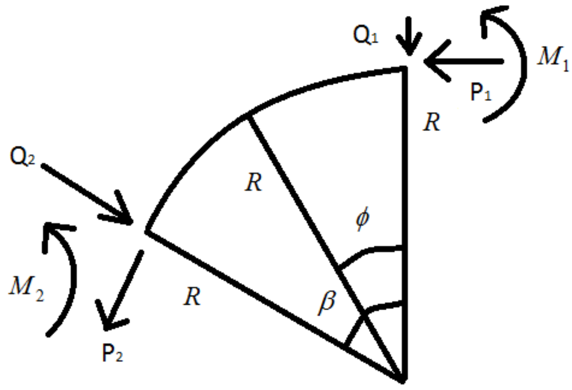
The strain energy of axially loaded bar is as given by

$$U = \frac{P^2 l}{2AE} \tag{2}$$

By taking the derivative of this expression with respect to P,

$$\frac{dU}{dP} = \frac{Pl}{AE} \tag{3}$$

The curved beam shown in the Fig.1 has the subtended angle β and radius = R force P_1 and P_2 are acting on node 1 and 2 respectively parallel to the direction of radius and Q_1 and Q_2 are the forces acting on node 1 and 2 respectively perpendicular to the direction of radius and M_1 and M_2 are the moments acting on node 1 and 2 respectively.



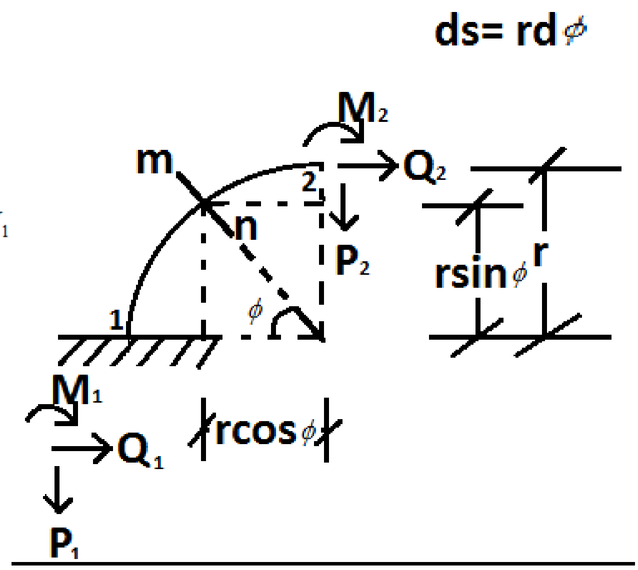


Figure 1: Curved beam section

Figure 2: Curved beam – Node 1 is fixed

Derivation of Element Stiffness Matrix

The moment at section m – n from Fig.2,

$$M = -P_2 r\cos\phi - Q_2 r(1-\sin\phi) - M_2 \tag{4}$$

The directions of the forces are shown in Fig. 2. By applying Castigliano's theorem on above equation (4), for various cases, to obtain displacement as well as rotations for all possible cases, following equations are obtained. Displacements and rotations at the node 1 for a special case of curved beam when $b = \dfrac{p}{2}$ are given as below.

$$\delta_{VP} = \frac{P_2 r^3}{EI_Z}\left[\frac{\beta}{2} - \frac{\sin 2\beta}{4}\right], \quad \delta_{VQ} = \frac{P_2 r^3}{EI_Z}\left[-\cos\beta + 1 - \frac{\sin^2\beta}{2}\right], \quad \delta_{VM} = -\frac{M_2 r^2}{EI_Z}\left[\cos\beta - 1\right]$$

$$\delta_{HP} = \frac{P_2 r^3}{EI_Z}\left[-\cos\beta + 1 - \frac{\sin^2\beta}{2}\right], \quad \delta_{HQ} = \frac{Q_2 r^3}{EI_Z}\left[\frac{3\beta}{2} - 2sin\beta + \frac{\sin 2\beta}{4}\right]$$

$$\delta_{HM} = \frac{Q_2 r^2}{EI_Z}\left[\sin\beta - \beta\right], \quad \theta_P = \frac{P_2 r^2}{EI_Z}\left[\sin\beta\right], \quad \theta_Q = -\frac{Q_2 r^2}{EI_Z}\left[\beta - \sin\beta\right], \quad \theta_M = \frac{M_2 r}{EI_Z}\beta \tag{5}$$

The above equations can be written in combined form. By applying principle of superposition, the equations can be written in matrix form as follows.

$$\begin{pmatrix} v_2 \\ u_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi r^3}{4EI_Z} & \frac{r^3}{2EI_Z} & \frac{r^2}{EI_Z} \\ \frac{r^3}{2EI_Z} & \frac{r^3}{6EI_Z}\left[\frac{3\pi}{4}-2\right] & \frac{r^2}{EI_Z}\left[\frac{\pi}{2}-1\right] \\ \frac{r^2}{EI_z} & \frac{r^2}{EI_Z}\left[\frac{\pi}{2}-1\right] & -\frac{\pi r}{2EI_z} \end{pmatrix} \begin{pmatrix} P_2 \\ Q_2 \\ M_2 \end{pmatrix} \tag{6}$$

Equilibrium equations from the free body diagram shown in Fig. 2 when 1^{st} is fixed and 2^{nd} is free[7]

$$\sum F_y = 0; Q_1 + Q_2 = 0; \boxed{Q_1 = -Q_2}$$
$$\sum F_x = 0; P_1 + P_2 = 0; \boxed{P_1 = -P_2}$$
$$\sum M_2 = 0; -M_2 - M_1 + P_1 r + Q_1 r = 0; \boxed{M_1 = -M_2 - P_2 r - Q_2 r} \tag{7}$$

Remaining quantities of forces are found from the equilibrium equations (7) and will be used in writing element stiffness matrix later.

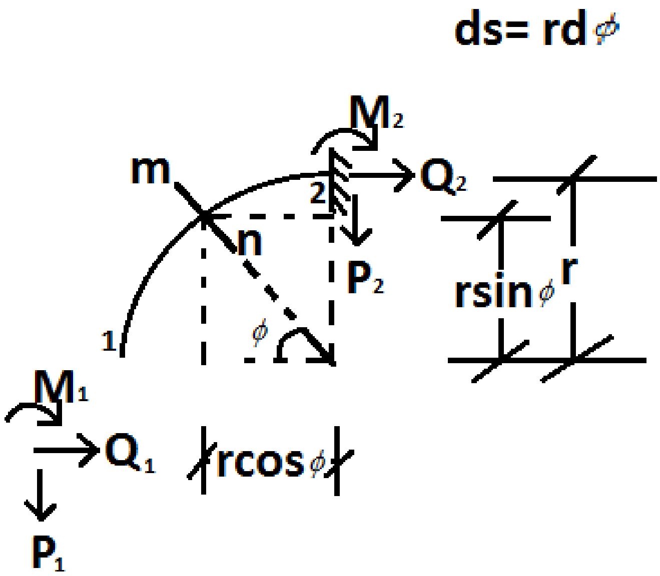


Figure 3: Curved beam – Node 2 is fixed

Moment at section m-n, referring to Fig. 3.

$$M = P_1 r(1-\cos\phi) + Q_1 \sin\phi - M_1 \tag{8}$$

By applying Castigliano's theorem on Equation (8), Displacement and rotation are obtained when node 2 is kept fixed and 1 is free.

$$\delta_{VP}=\frac{P_1r^3}{EI_Z}\left[\frac{3\beta}{2}-2sin\beta+\frac{\sin 2\beta}{4}\right],\ \delta_{VQ}=\frac{Q_1r^3}{EI_Z}\left[-\cos\beta+1-\frac{\sin^2\beta}{2}\right],\ \delta_{VM}=-\frac{M_1r^2}{EI_Z}\left[\beta-\sin\beta\right]$$

$$\delta_{HP}=\frac{P_1r^3}{EI_Z}\left[-\cos\beta+1-\frac{\sin^2\beta}{2}\right],\ \delta_{HP}=\frac{Q_1r^3}{EI_Z}\left[\frac{\beta}{2}-\frac{\sin 2\beta}{4}\right],\ \delta_{HM}=\frac{M_1r^2}{EI_Z}\left[\cos\beta-1\right]$$

$$\theta_{P1}=-\frac{P_1r^2}{EI_Z}\left[\beta-\sin\beta\right],\ \theta_{Q1}=\frac{Q_1r^2}{EI_Z}\left[\cos\beta-1\right],\ \theta_{M1}=\frac{M_1r}{EI_Z}\beta \tag{9}$$

Equilibrium equations when 1st is free and 2nd is fixed are written as follows.

$$\begin{aligned}&\sum F_y=0;Q_1+Q_2=0;\boxed{Q_2=-Q_1}\\&\sum F_x=0;P_1+P_2=0;\boxed{P_2=-P_1}\\&\sum M_2=0;-M_2-M_1+P_1r+Q_1r=0;\boxed{M_2=-M_1+P_1r+Q_1r}\end{aligned} \tag{10}$$

Eq. (9) can be combined after applying superposition principle and written in matrix form as follows, when node of the curved beam 1 is kept free and 2 is fixed.

$$\begin{pmatrix}v_1\\u_1\\\theta_1\end{pmatrix}=\begin{pmatrix}\frac{r^3}{EI_Z}\left[\frac{3\pi}{4}-2\right] & \frac{r^3}{2EI_Z} & -\frac{r^2}{EI_Z}\left[\frac{\pi}{2}-1\right]\\ \frac{r^3}{2EI_Z} & \frac{\pi r^3}{4EI_Z} & -\frac{r^2}{EI_Z}\\ -\frac{r^2}{EI_z}\left[\frac{\pi}{2}-1\right] & -\frac{r^2}{EI_Z} & \frac{\pi r}{2EI_z}\end{pmatrix}\begin{pmatrix}P_1\\Q_1\\M_1\end{pmatrix} \tag{11}$$

By taking help of Eqs. (6), (7), (10), (11), the final element stiffness matrix is written as Eq. (12), which can later be used for solving numerical examples.

$$\begin{bmatrix} P_1 \\ Q_1 \\ M_1 \\ P_2 \\ Q_2 \\ M_2 \end{bmatrix} = \frac{r^4}{\left(EI_z\right)^2} \begin{bmatrix} \left[\frac{\pi^2}{8}-1\right] & \left[\frac{\pi}{4}-1\right] & \left[\frac{r}{8}\left[\pi^2-2\pi-4\right]\right] & -\left[\frac{\pi^2}{8}-1\right] & -\left[\frac{\pi}{4}-1\right] & -\frac{r}{8}\left[\pi^2-2\pi-4\right] \\ \left[\frac{\pi}{4}-1\right] & \left[\frac{\pi^2}{8}-1\right] & \frac{r}{2}\left[\pi-3\right] & -\left[\frac{\pi}{4}-1\right] & -\left[\frac{\pi^2}{8}-1\right] & -\frac{r}{2}\left[\pi-3\right] \\ \left[\frac{r}{8}\left[\pi^2-2\pi-4\right]\right] & \frac{r}{2}\left[\pi-3\right] & \left[\frac{r^2}{16}\left[3\pi^2-8\pi-4\right]\right] & \left[\frac{1}{8}\left[\pi^2+2\pi-16-r\left[\pi^2-2\pi-4\right]\right]\right] & \left[\frac{1}{8}\left[\pi^2+2\pi-16-4r\left[\pi-3\right]\right]\right] & \left[\frac{r}{8}\left[\pi^2+2\pi-16-\frac{r}{2}\left[3\pi^2-8\pi-4\right]\right]\right] \\ -\left[\frac{\pi^2}{8}-1\right] & -\left[\frac{\pi}{4}-1\right] & -\frac{r}{2}\left[\pi-3\right] & \left[\frac{\pi^2}{8}-1\right] & \left[\frac{\pi}{4}-1\right] & \frac{r}{2}\left[\pi-3\right] \\ -\left[\frac{\pi}{4}-1\right] & -\left[\frac{\pi^2}{8}-1\right] & \left[\frac{r}{8}\left[\pi^2-2\pi-4\right]\right] & \left[\frac{\pi}{4}-1\right] & \left[\frac{\pi^2}{8}-1\right] & -\left[\frac{r}{8}\left[\pi^2-2\pi-4\right]\right] \\ \left[-\frac{r}{8}\left[\pi^2-2\pi-4\right]\right] & -\frac{r}{2}\left[\pi-3\right] & \left[\frac{r}{8}\left[\pi^2+2\pi-16-\frac{r}{2}\left[3\pi^2-8\pi-4\right]\right]\right] & \frac{r}{2}\left[\pi-3\right] & -\frac{r}{8}\left[\pi^2-2\pi-4\right] & \frac{r^2}{16}\left[3\pi^2-8\pi-4\right] \end{bmatrix} \begin{bmatrix} v_1 \\ u_1 \\ \theta_1 \\ v_2 \\ u_2 \\ \theta_2 \end{bmatrix}$$

$$\frac{r^7}{\left(EI_z\right)^3}\left[\frac{\pi^3}{32}-\frac{5\pi}{2}+1\right] \tag{12}$$

Above is the element stiffness matrix obtained.

By writing the Value of β as 90^o and also solving the above equation we obtained the element stiffness matrix for a special case $\beta = 90^o$.

$$\begin{bmatrix} P_1 \\ Q_1 \\ M_1 \\ P_2 \\ Q_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 42.87 & -39.37 & -9.48r & -42.87 & 39.37 & 9.48r \\ -39.37 & 42.87 & 12.98r & 39.37 & -42.87 & -12.98r \\ -9.48r & 12.98r & 5.45r^2 & 12.98r & -9.48 & -1.95r^2 \\ -42.87 & 39.37 & 12.98r & 42.87 & -39.37 & -12.98r \\ 39.37 & -42.87 & -9.48r & -39.37 & 42.87 & 9.48r \\ 9.48 & -12.98r & -1.95r^2 & -12.98r & 9.48r & 5.45r^2 \end{bmatrix} \begin{bmatrix} v_2 \\ u_2 \\ \theta_2 \\ v_2 \\ u_2 \\ \theta_2 \end{bmatrix} \tag{13}$$

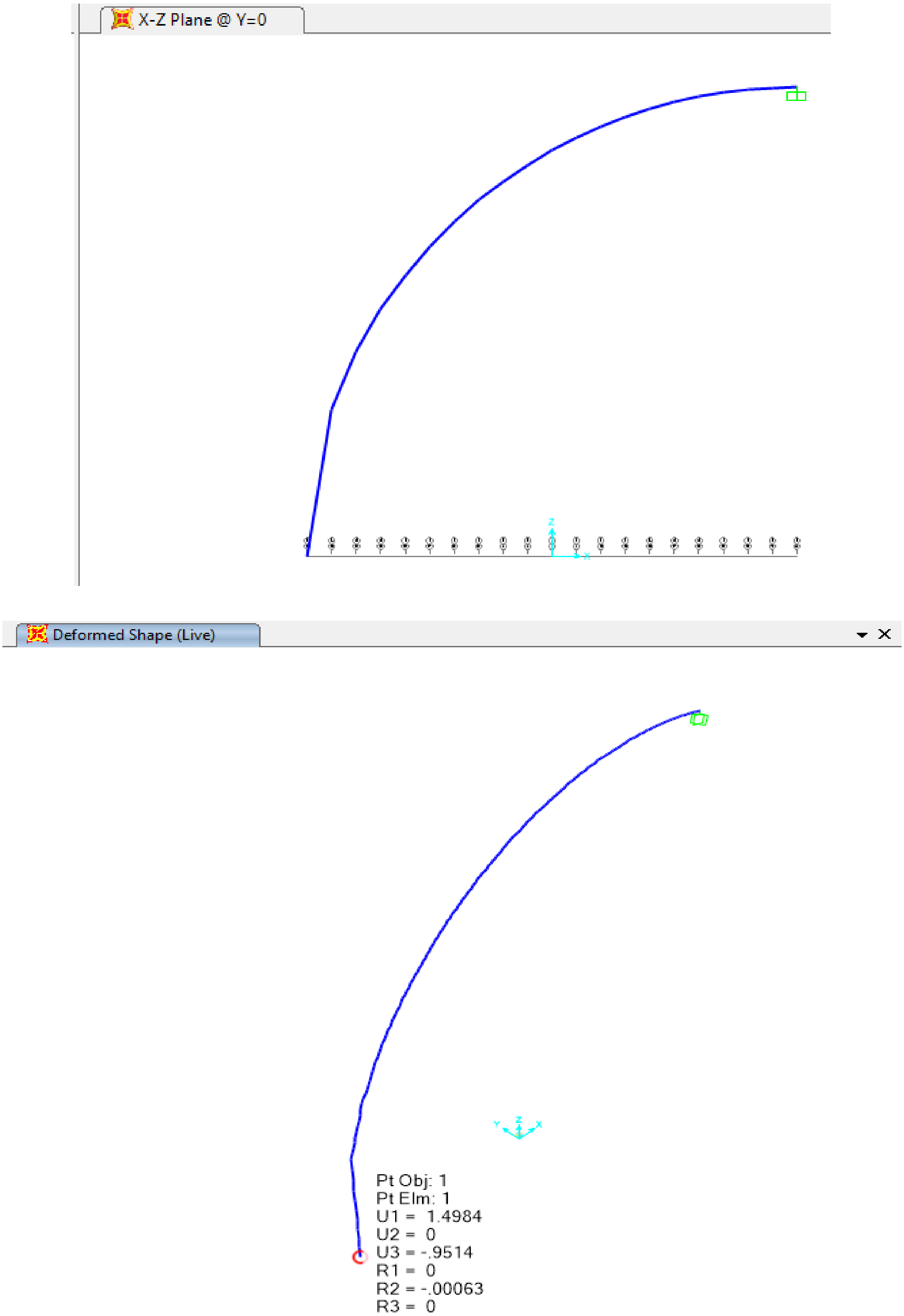


Figure 4: Model in SAP2000

Numerical Example

By analysing simple numerical example, equations for stiffness matrix are validated. A curved beam having radius of curvature 3000 mm and cross section of 500x500 mm,M30 and modulus of elasticity(E)= $5000\sqrt{f_{ck}}$. M30 is the grade of Concrete, f_{ck} is the characteristic strength of concrete.

Result through current direct stiffness matrix, the displacement in horizontal direction at the end point=1.4867 mm. Displacement in downward direction at the end point = 0.946 mm. Rotation at he end point = -0.00063(R). The curved beam model is generated in SAP2000 as shown in Fig. 4. Fig. 4 shows the curved beam exmaple which free at node 1 and supported at node 2. The horizontal and vertical loading has been applied at node 1 of unit magnitude. Here in Fig. 4, U_1 and U_3 are the deformations in horizontal and vertical direction. R_2 is the roation about the axis which out of plane of the figure.

Results from the software SAP200 are given as: The displacement in horizontal direction at the end point = 1.498 mm; Displacement in downward direction at the end point = -0.9514mm; Rotation at he end point = -0.00063(R).

Conclusion

An element stiffness matrix has been derived by using of strain energy concept and knowledge of basic mechanics. Two nodes of the curved beam element having 3 degrees of freedom, (1) Moment, (2) Vertical displacement and (3) horizontal displacement. The obtained final element stiffness matrix will be useful for solving variety of curved beam examples applied in practice such as arches, hooks, and curved traffic poles subjected to various loading conditions. Numerical example has been demonstrated for the validity of the equations and compared with the software results. The problem is analyzed using direct stiffness approach.

Nomenclature

U	=	Strain Energy of the elastic body
AE	=	Axial Rigidity
M	=	Moment

δ	=	Displacement
θ	=	Rotation
M_1, M_2	=	Moment occurred at node 1 and 2 respectively
Q_1, Q_2	=	Forces acting parallel to the axis of the curve beam
ds	=	Arc length
R	=	Radius of curvature
r	=	Radial coordinate along the curve
$d\varphi$	=	Subtended angle
P_1, P_2	=	Normal forces acting at node 1 and node 2 respectively
β	=	Total angle along the curve
δ_{VP}	=	Displacement in vertical direction due to load P
δ_{HP}	=	Displacement in Horizontal direction due to load P
δ_{VQ}	=	Displacement in vertical direction due to load Q
δ_{HQ}	=	Displacement in Horizontal direction due to load Q
δ_{VM}	=	Displacement in vertical direction due to Moment M
δ_{HM}	=	Displacement in Horizontal direction due to Moment M
v_1, v_2	=	Displacement in normal direction to the axis of curved beam at node 1 and 2
u_1, u_2	=	Displacement in perpendicular direction to the axis of curved beam at node 1 and 2
θ_1, θ_2	=	Rotations at node 1 and 2
θ_{P1}, θ_{P2}	=	Rotations at node 1 and 2 due to load p
θ_{Q1}, θ_{Q2}	=	Rotations at node 1 and 2 due to load Q
θ_{M1}, θ_{M2}	=	Rotations at node 1 and 2 due to moment M
P_1, P_2	=	Loads at node 1 and 2 in normal direction
Q_1, Q_2	=	Loads at node 1 and 2 in parallel direction
M_1, M_2	=	Moments at node 1 and 2

References

1. Lee, H. P. (1969). Generalized stiffness matrix of a curved – beam element. *AIAA Journal, 7*(10), 2043 – 2045.
2. Li, X., Zhao, Y., Zhu, C., Chen, C. (2012). Exact solutions of stiffness matrix for curved beams with pinned – pinned ends, *Advanced Materials Research*, (Vols. *368 – 373)*, 3117 – 3120.
3. Zhang, C. (1998). *A curved beam element and its application to traffic poles*. (Master's thesis, University of Manitoba, Manitoba, Canada). Retrieved from https://www.collectionscanada.gc.ca/obj/s4/f2/dsk2/ftp01/MQ32291.pdf
4. Palaninathan, R., Chandrasekharan, P. S. (1985). Curved beam element stiffness matrix formulations, *Computers and Structures*, *21*(4), 663 – 669.
5. Timoshenko, S. P. (2004). *Strength of Materials*. Vol. I, (3rd ed.). New Delhi: CBS publishers & Distributors.
6. Swaroop, A. (2014). *Mechanics of Materials*. (2nd ed.). New Delhi: New Age International Private Limited.
7. Martin H. C. (1966). *Introduction to Matrix methods of structural analysis*. New York : McGraw-Hill.

