Construction of Fibonacci Spiral and Geometry in Golden Hexagon using Golden Sections

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Abstract

In this article, various geometry and construction of Fibonacci spiral is drawn in golden hexagon using golden section. Mathematical properties of the constructed geometry may be investigated. The above is the novel work of the present article. The article gives introduction to golden ratio and golden spiral. For beginners, as instruction guide for construction of various geometrical basics shapes such as, golden rectangle, golden triangle, golden rectangle spiral and golden triangle spiral, golden pentagon and golden hexagon is given. Article includes also the couple of examples in nature, paintings, and sculptures with superimposed images which are correlated with present geometrical construction of basic shapes.

Keywords

Golden section, Golden hexagon, Golden ratio

Introduction

Golden ratio is an irrational number that's equal to approximately 1.6180 and is written by Greek letter φ. When we divide a line into two parts such that the whole length is divided by the long part is also equal to the long part divided by the short part[1].

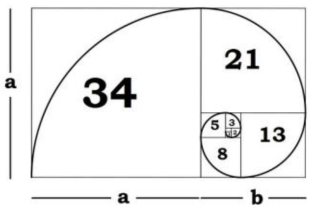

Figure 1: Golden Line **Figure 2: Golden Rectangle** **Figure 3: The Great Pyramid**

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Here,

$$\frac{AB}{AC} = \frac{AC}{CB} \cong 1.6180 \quad (1)$$

Artists and architects believe that the golden ratio makes the most pleasing and beautiful shape.

Besides being beautiful the resulting shape has interesting characteristics.

The golden number can be applied to the proportions of a rectangle, called the golden rectangle, other basic geometric shapes such as triangles; pentagons; hexagons etc… This is known as one of the most visually satisfying of all geometric forms. Hence,the appearance of the Golden Ratio in Art. If you draw a golden rectangle, and then draw line inside it to divide that rectangle into a square and another smaller rectangle, that smaller rectangle will amazingly be another golden rectangle. You can do this again with this new golden rectangle, and you will get once again a square and yet another golden rectangle (Fig. 2). This process can continue till infinite. Mathematically, this property is visualized in following equation in terms of continued fraction[2],

$$\varphi = 1 + \frac{1}{\varphi} \quad \text{or} \quad \varphi^2 - \varphi - 1 = 0 \quad (2)$$

Here,

$$\frac{a}{b} = \frac{(a+b)}{a} \cong 1.6180 = \varphi \quad (3)$$

Many buildings and artworks have the golden ration in them, such as The Great Pyramid, the Parthenon in Greece, but it is not really known if it was designed that way. In Great Pyramid of Giza, (Fig. 3), the length of each side of the base is 756 feet with a height of 481 feet. The ratio of the base to the height is roughly 1.5717, which is close to the Golden Ratio. Leonardo da Vinci used the Golden ratio to define all of the proportions in his creations.

Around, 1200, mathematician Leonardo Fibonacci discovered the unique properties of Fibonacci sequence. This sequence ties directly into the Golden Ratio.

Fibonacci sequence is the series of number 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

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The next number in the Fibonacci sequence is found by adding up the two numbers before it. Here, connection between the Golden ratio and the Fibonacci sequence is given by dividing each number in the Fibonacci sequence by the previous number in the sequence, gives: 1/1 = 1, 2/1 =, 3/2 = 1.5, and 144/89 = 1.6179...., number reaches a closer approximation of the golden ratio φ and getting closer and closer to 1.6180[3].

The Golden and Fibonacci Spiral

The celebrated golden spiral is a special case of the more general logarithmic spiral whose radius r is given by[4]

$$r = ae^{b\theta} \tag{4}$$

Where θ is the usual polar angle, and a and b are constants. Jacob Bernoulli (1655 – 1705) studies this spiral in depth and gave it the name spira mirabilis, or miraculous spiral. The golden spiral is a logarithmic spiral whose radius either increases or decreases by a factor of the golden ratio φ with each one – quarter turn, that is, when θ increases by $\frac{\pi}{2}$.

The golden spiral therefore satisfies the equation

$$r = a\varphi^{\frac{2\theta}{\pi}} \tag{5}$$

In fig. 2, within the golden rectangle, the dimension of each succeeding square decreases by a factor of φ, with four squares composing each quarter turn of the spiral. It is then possible to inscribe a golden spiral within the figure of golden rectangle with spiralling squares. The central point of the spiral at the accumulation point of all the squares, and fit the parameter a so that the golden spiral passes through opposite corners of the squares.

A Fibonacci spiral approximates the golden spiral using quarter – circle arcs inscribed in squares of integer Fibonacci – number side, shown for square sizes 1, 1, 2, 3, 5, 8, 13, 21, and 34. The resulting Fibonacci spiral is shown in Fig. 2.

Geometrical Construction of Golden Rectangle, Golden Triangle, Golden Pentagon and Golden Hexagon

In this section, the geometrical construction of Golden Rectangle is described[5].

Geometrical Construction of Golden Rectangle

A golden rectangle is a rectangle with side lengths that are in the golden ratio (about 1:1.618). This section also explains how to construct a square, which is needed to construct a golden rectangle.[5]Step 1: Draw a square. Let us name the vertices of the square as A, B, C and D.

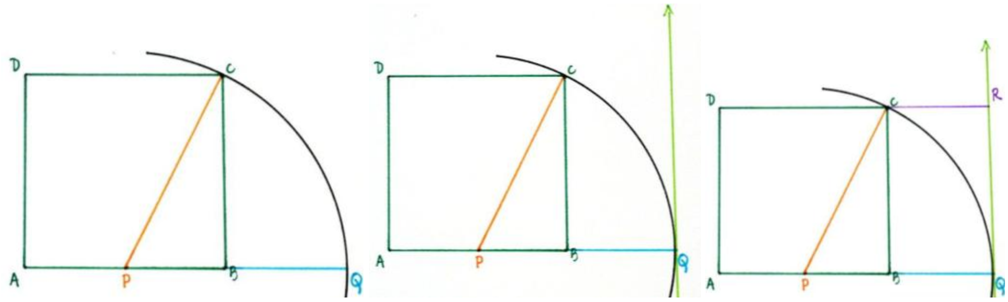

Figure 4: (a, b, c) Golden Rectangle

Step 2: Locate the mid-point of any one side of the square by bisecting it. Let us pick the side AB and call its mid-point as point P. Step 3: Connect the mid-point P to a corner of the opposite side. Since P lies on the side AB, the opposite side shall be the side CD. Let's choose to connect P with C. Step 4: Place the tip of the compass on P and set its width to match the distance PC. Draw a large arc towards the side BC. Step 5: Extend the side AB to cut the arc at some point (say Q).Step 6: Draw a line parallel to the side BC, passing through the point Q. Step 7: Extend the side DC to meet the parallel line at some point (say R). Step 8: Erase any extraneous constructions if you so wish. You may verify that the ratio of the measure of the shorter side of the rectangle (QR or AD) to the measure of its longer side (AQ or RD) is very close to 1:1.618. Further the rectangle CRQB is another golden rectangle, in which another square is made of length BQ gives us third golden rectangle inside CRQB and the process continues. (Fig. 4 (a, b, c)).

Golden line, rectangle and spiral Examples

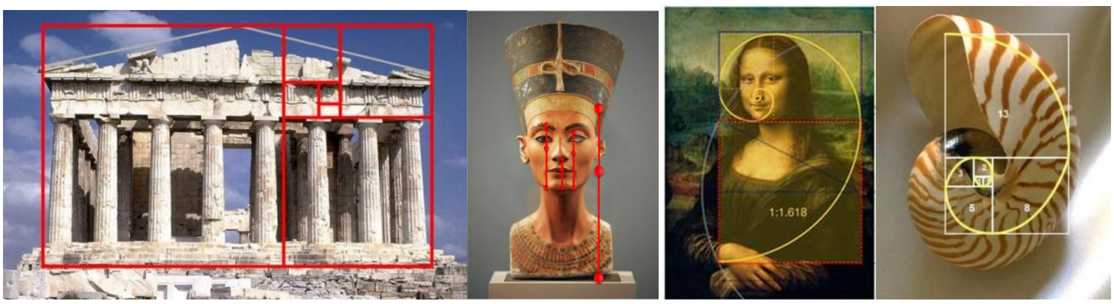

Figure: 5 (a, b, c, d) Golden Ratio in real life

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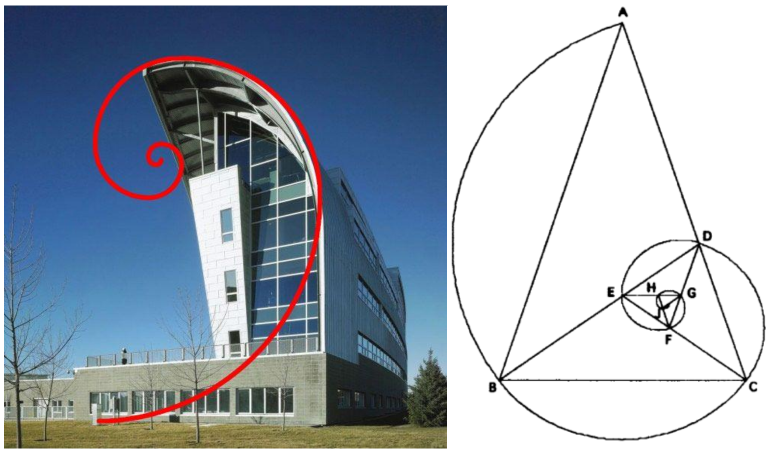
Fig. 5a shows the famous monument Parthenon in Greece,the golden section used in the Ancient Egypt sculpture is seen in Fig. 5b.

Construction of Golden Rectangle Spiral[7]

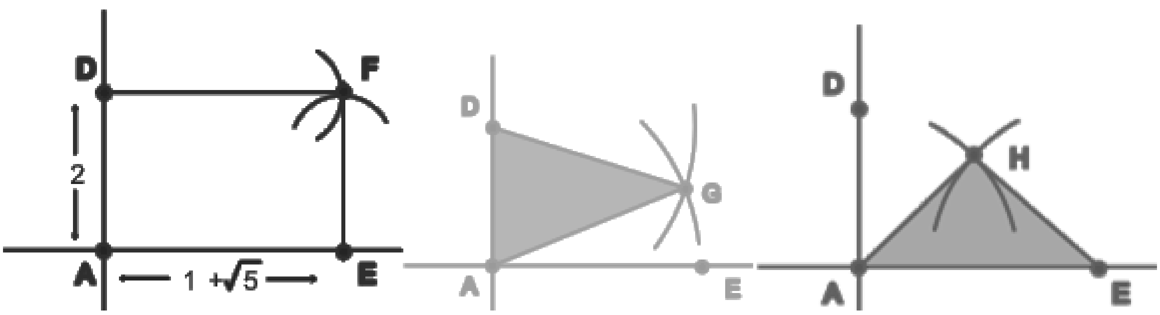
To draw a golden spiral, consider the centre of the square C and radius CD for the one turn of the spiral. Similarly, various centres and radius are obtained for all squares and using quarter – circle arcs inscribed in squares, the spiral is completed as shown in Fig. 2. Golden section, rectangle and spiral is seen in paintings, sculptures, building, nature (egg shell), human DNA molecule etc…Fig. 5c, 5d - f.



(e) (f)

Figure 5: Golden rectangles and triangle spiral

Construction of Golden Triangle and Golden Triangle Spiral



(a) **(b)** **(c)**

Figure 6: Construction of golden triangle

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Draw golden rectangle using the method as described in above section. As shown in Fig. 6a, the golden rectangle has longer side as (1 + $\sqrt{5}$ units) and shorter side as 2 units. Draw twointersecting arcs of radius the length of *AE*. One with compass point at point *A* and the other with compass point at point *D*. Label the point of intersection *G*. The triangle *ADG* is a golden triangle; it is an isosceles triangle (Fig. 5f) with the ratio of the longer side (1 + $\sqrt{5}$ units) to the shorter side (2 units) equalling the golden ratio.(Fig.6b).[6]Alternatively you may construct a golden triangle by drawing two arcs of radius length*AD*, one centred at point *A* and the other centred at point *E*. Label the point of intersection of the arcs *H*. (Fig. 6c).Smaller triangles inside the big golden triangle are obtained by bisecting the angles for example, angles B and C, angles D and C, E and D, E and F etc… as shown in Fig. 5(f). For drawing the spiral in golden triangles, draw and arc of AB by considering the circle centre at D, draw arc BC considering the circle centre, draw arc CD by considering the circle centre at F, draw arc DE by considering the circle centre at G, draw arc EF by considering the circle centre at H, draw an arc of FG by considering circle centre at J (Fig.5f).[7]

Construction of Golden Pentagon

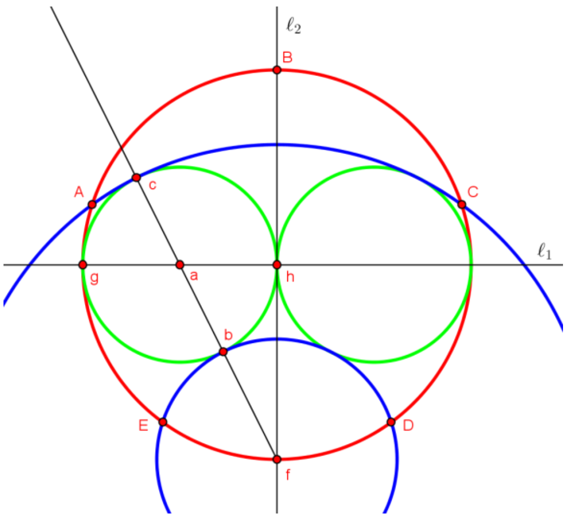
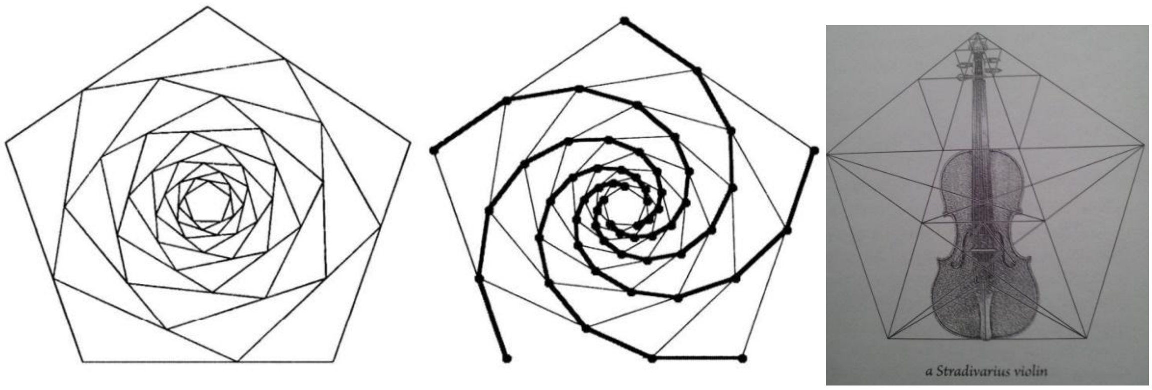


Figure 7a: Golden Pentagon

Constructing golden pentagon comes from Yosifusa Hirano of 19th Century Japan. It is elegant method of constructing the pentagon.[8] The following are the steps.1) Draw a circle (the red one) with center hh.2) Draw the perpendicular lines ℓ1ℓ1 and ℓ2ℓ2 through hh. Locate the points of intersection ff, BB, and ggwith the red circle.3) Bisect the line segment ghgh. Denote the center by aa.4) Draw the green circle with center aa and radius ahah.5) Draw the other green circle (as in steps (3) and 4)).6) Draw the line segment through ff and aa.7) Locate the points of intersection bb and cc of the line segment with the circle constructed in step 4).8) Draw the blue arcs (both have center at ff and the radii are fbfb and fcfc).9) Locate the points of intersection AA, CC, DD, and EE.



(b) (c) (d)

Figure 7bcd: Golden Pentagon

Fig. (7b-c) shows a design made from golden pentagon as well as it is correlated with musical instrument in fig. 7d.

Construction of Golden Hexagon

Consider an equilateral triangle ABC as shown in Fig. 8. Let the points E, D and G on each sides of the triangle such as

$$\frac{CE}{EA} = \frac{BD}{DC} = \frac{AG}{GB} = \varphi \quad (1)$$

Construct triangle EDG inside triangle ABC.In triangle EDG, Obtain points I, j and k in such a way EI = ID, EJ=JG, DK=KG.Also, draw line passing through vertex C and a point I intersects at line AB at point M, similarly, draw line passing through vertex B and point k which interacts line AC at point **K.** Similarly, also draw line passing through the vertex A and point j which intersects line CD at L. (Fig. 8) Construct hexagon KDLGMEK (Fig. 8) for which

$$\frac{KD}{KE} = \frac{EM}{MG} = \frac{GL}{LD} = \phi .$$

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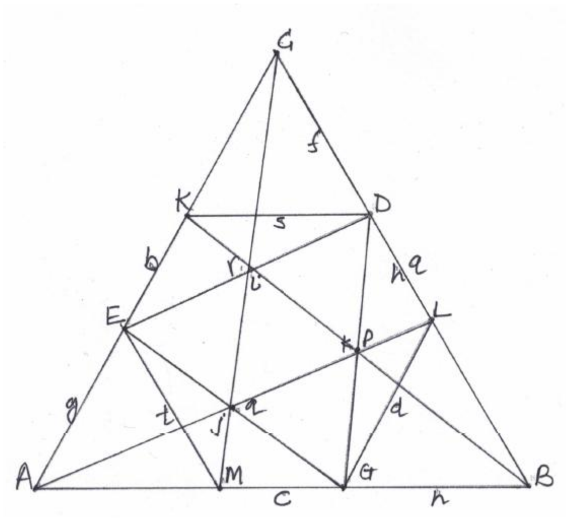

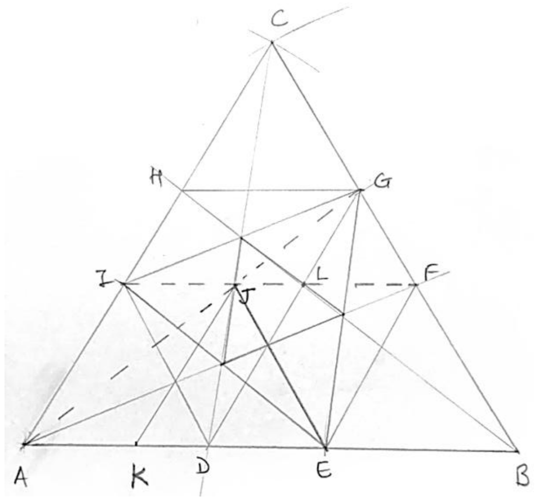

Figure 8: Construction of Hexagon **Figure 9: Golden section in golden hexagon**

Construction of Geometry

Construct golden hexagon DEFGHI inside equilateral triangle ABC as shown in Fig. 9.Join IF to make parallelogram of AEFI.The lines AG, CD and IF meet at J, where now IJDE completes another parallelogram inside larger parallelogram AEFI. Join line JE to complete the smaller parallelogram DEJI.

On line AD, obtain point K in such a way $\frac{AK}{KD} = \varphi = \frac{AD}{AK}$.

Join JK such that JKE gives one triangle. Join DG and HB and intersection of these lines with IF meets at L where

$\frac{FL}{LJ} = \varphi = \frac{FJ}{FL}$ and EJF completes the triangle.

Here, naturally the intersection of lines IF and AG intersection at J, wherein $\frac{IJ}{JL} = \varphi = \frac{IL}{IJ}$.

This is the base line of the triangle DLI.

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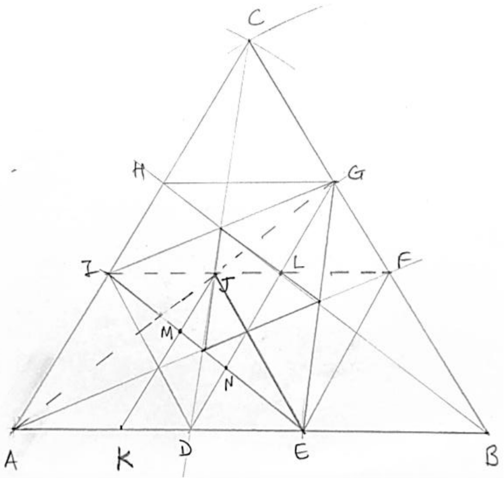
Construction of Spiral

Intersection of lines KJ from triangle KJE, The diagonal line of parallelogram DEJI, gives point M,Similarly intersection of lineDL from triangle DLI and the diagonal IE gives N.(Fig. 10)



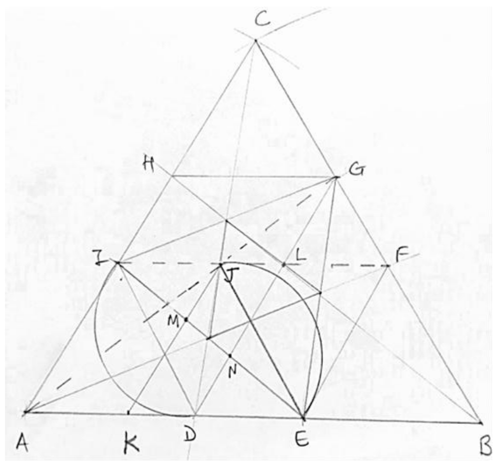


Figure 10: Marking golden sections **Figure 11: Obtaining locations and sections**

Complete the part of spiral of arc length MI from M covering the side ID of triangle ADI and triangle DLI.Complete the part of spiral of arc length NE from N covering the side JE of triangle EFJ and KEJ (Fig. 11).

From larger parallelogram, obtain intersection of lines AF and line JE of triangle EFJ as well as intersection of lines ID and AF meets at points O and P respectively. Complete the spiral of arc length OF and AP covering sides FI and AE. Here, it is possible to imagine a larger triangle AEH as well as triangle of base length IF wherein virtual point of the triangle can be imagined H' passing through the lines ID and FE.

Here, it can be observed from Fig. 12 that $\frac{JP}{JM} = \frac{JM}{MP} = \varphi, \ \frac{DO}{DN} = \frac{DN}{NO} = \varphi$.

Now Imagine a parallelogram DOJP and NOMP.

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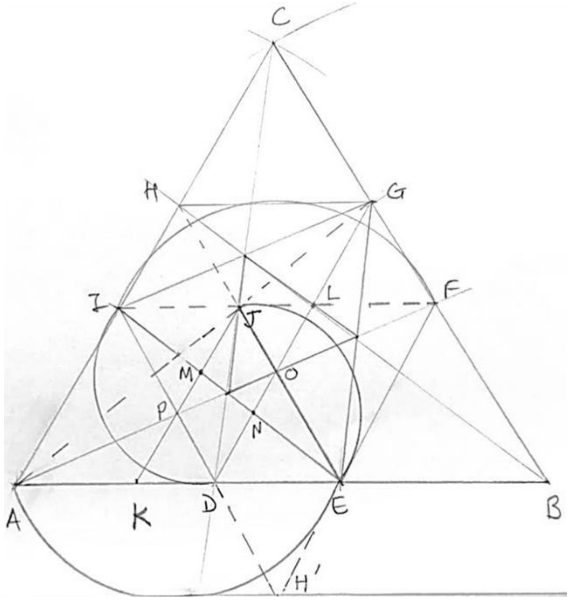

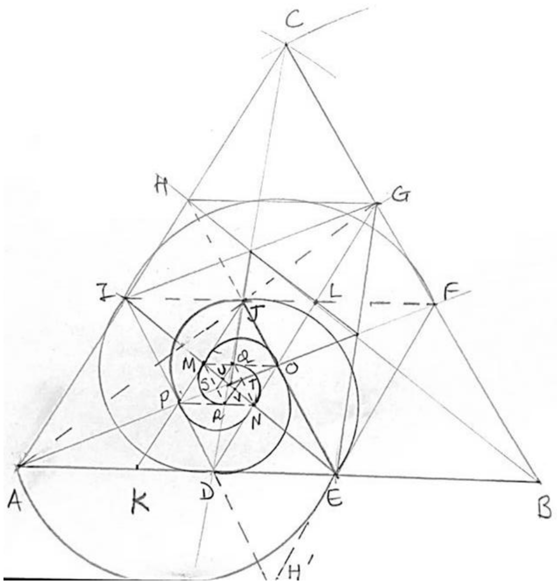

Figure 12: Construction of Golden Spiral Figure 13: Complete Spiral in golden hexagon

Parallelogram DOJP

Consider DOJP. Connect MO and PN to obtain triangle MOJ and DNP. Intersection of line DC cuts the triangle of line MO and PN to obtain points Q and R respectively. Line JD is a diagonal of parallelogram DOJP.In parallelogram, DOJP, draw an arc of length QJ from Q and RD from R covering the triangle PJI of side PJ and triangle DOE of sides DO (Fig. 13).

ParallelogramNOMP

Consider PNOM which has diagonal PO. Complete the triangle PRM and NOQ by connection M and R and QN to obtain lines MR and QN which intersects diagonal POat S and T respectively. Complete the arc of length TO and PS from T and S respectively covering the sides MO of triangle MOJ and PN of triangle DNP (Fig. 13).

Parallelogram RNQM

Consider RNQM which has the diagonal MN. Complete MSQ and RTN triangle in such a way one of the sides of the triangle intersects MN at U and V respectively.Complete the arc of length MU and NV from U and V covering the sides of the triangle PMR and QON. This completes the spiral (Fig. 13).Since golden ratio has the continued and infinite properties, it is true for golden hexagon and parallelogram inside golden hexagon follows this property of continued and infinite ratios.

Conclusion

Systematic construction steps of geometry and Fibonacci spiral has been presented in this article. The mathematical and geometrical properties may be investigated using these constructions. The design presented in this article is useful for artists, architects and mathematicians for further exploration. The method is applicable at any scale, micro to macro, but in any and all scales construction, the golden ratio whereever appears remains the same and constant. It is a general method, not for specific case such as 8/5=1.6180. This method is valid for any construction parameters of geometry. Irregular golden section hexagon is not common geometrical shape and has not been used in building proportions study. The reason for this is its difficult construction unlike many other well known basic shapes such as golden rectangles and golden triangles. The designer can think of a design by incorporating this shape in their construction including the spiral in golden section either in the form of symmetry and where optimized shape becomes the necessity for saving the space. A shape presented here can be a cross section of building and three dimensional objects. One such situation is occurred in platonic solids, where 4 vertices of icosahedron is golden rectangle.

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